

A STEENROD SQUARE ON KHOVANOV HOMOLOGY

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ABSTRACT. In a previous paper, we defined a space-level version $\mathcal{X}_{Kh}(L)$ of Khovanov homology. This induces an action of the Steenrod algebra on Khovanov homology. In this paper, we describe the first interesting operation, $Sq^2: Kh^{i,j}(L) \rightarrow Kh^{i+2,j}(L)$. We compute this operation for all links up to 11 crossings; this, in turn, determines the stable homotopy type of $\mathcal{X}_{Kh}(L)$ for all such links.

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1. INTRODUCTION

Khovanov homology, a categorification of the Jones polynomial, associates a bigraded abelian group $Kh_{\mathbb{Z}}^{i,j}(L)$ to each link $L \subset S^3$ [Kho00]. In [LS] we gave a space-level version of Khovanov homology. That is, to each link L we associated stable spaces (finite suspension spectra) $\mathcal{X}_{Kh}^j(L)$, well-defined up to stable homotopy equivalence, so that the reduced cohomology $\tilde{H}^i(\mathcal{X}_{Kh}^j(L))$ of these spaces is the Khovanov homology $Kh_{\mathbb{Z}}^{i,j}(L)$ of L . Another construction of such spaces has been given by [HKK].

The space $\mathcal{X}_{Kh}^j(L)$ gives algebraic structures on Khovanov homology which are not (yet) apparent from other perspectives. Specifically, while the cohomology of a spectrum does not have a cup product, it does carry stable cohomology operations. The bulk of this paper is devoted to giving an explicit description of the Steenrod square

$$Sq^2: Kh_{\mathbb{F}_2}^{i,j}(L) \rightarrow Kh_{\mathbb{F}_2}^{i+2,j}(L)$$

induced by the spectrum $\mathcal{X}_{Kh}^j(L)$. First we give a combinatorial definition of this operation Sq^2 in Section 2 and then prove that it agrees with the Steenrod square coming from $\mathcal{X}_{Kh}^j(L)$ in Section 3. (For the reader's convenience, the algorithm is summarized, with an example, in Appendix A.)

The description is suitable for computer computation, and we have implemented it in Sage. The results for links with 11 or fewer crossings are given in Section 5. In particular, the operation Sq^2 is nontrivial for many links, such as the torus knot $T_{3,4}$. This implies a nontriviality result for the Khovanov space:

Theorem 1. *The Khovanov homotopy type $\mathcal{X}_{Kh}^{11}(T_{3,4})$ is not a wedge sum of Moore spaces.*

Even simpler than Sq^2 is the operation $Sq^1: Kh_{\mathbb{F}_2}^{i,j}(L) \rightarrow Kh_{\mathbb{F}_2}^{i+1,j}(L)$. Let $\beta: Kh_{\mathbb{F}_2}^{i,j}(L) \rightarrow Kh_{\mathbb{Z}}^{i+1,j}(L)$ be the Bockstein homomorphism, and $r: Kh_{\mathbb{Z}}^{i,j}(L) \rightarrow Kh_{\mathbb{F}_2}^{i,j}(L)$ be the reduction mod 2. Then $Sq^1 = r\beta$, and is thus determined by the integral Khovanov homology; see also Subsection 2.5. As we will discuss in Section 4, the operations Sq^1 and Sq^2 together determine the Khovanov homotopy type $\mathcal{X}_{Kh}(L)$ whenever the Khovanov homology of L has a sufficiently simple form. In particular, they determine $\mathcal{X}_{Kh}(L)$ for any link L of 11 or fewer crossings; these homotopy types are listed in Table 1 (Section 5).

The subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2 is

$$\mathcal{A}(1) = \frac{\mathbb{F}_2\{Sq^1, Sq^2\}}{(Sq^1)^2, (Sq^2)^2 + Sq^1 Sq^2 Sq^1}$$

where $\mathbb{F}_2\{Sq^1, Sq^2\}$ is the non-commuting extension of \mathbb{F}_2 by the variables Sq^1 and Sq^2 . By the Adem relations, the next Steenrod square Sq^3 is determined by Sq^1 and Sq^2 , viz. $Sq^3 = Sq^1 Sq^2$. Therefore, the next interesting Steenrod square to compute would be $Sq^4: Kh_{\mathbb{F}_2}^{i,j}(L) \rightarrow Kh_{\mathbb{F}_2}^{i+4,j}(L)$.

The Bockstein β and the operation Sq^2 are sometimes enough to compute Khovanov K -theory: in the Atiyah-Hirzebruch spectral sequence for K -theory, the d_2 differential is zero and the d_3 differential is the integral lift $\beta \text{Sq}^2 r$ of Sq^3 (see, for instance [Ada95, Proposition 16.6] or [Law]). For grading reasons, this operation vanishes for links with 11 or fewer crossings; indeed, the Atiyah-Hirzebruch spectral sequence degenerates in these cases, and the Khovanov K -theory is just the tensor product of the Khovanov homology and $K^*(\text{pt})$. In principle, however, the techniques of this paper could be used to compute Khovanov K -theory in some interesting cases. Similarly, in certain situations, the Adams spectral sequence may be used to compute the real connective Khovanov KO -theory using merely the module structure of Khovanov homology $Kh_{\mathbb{F}_2}(L)$ over $\mathcal{A}(1)$.

2. THE ANSWER

2.1. Sign and frame assignments on the cube. Consider the n -dimensional cube $\mathcal{C}(n) = [0, 1]^n$, equipped with the natural CW complex structure. For a vertex $v = (v_1, \dots, v_n) \in \{0, 1\}^n$, let $|v| = \sum_i v_i$ denote the Manhattan norm of v . For vertices u, v , declare $v \leq u$ if for all i , $v_i \leq u_i$; if $v \leq u$ and $|u - v| = k$, we write $v \leq_k u$.

For a pair of vertices $v \leq_k u$, let $\mathcal{C}_{u,v} = \{x \in [0, 1]^n \mid \forall i: v_i \leq x_i \leq u_i\}$ denote the corresponding k -cell of $\mathcal{C}(n)$. Let $C^*(\mathcal{C}(n), \mathbb{F}_2)$ denote the cellular cochain complex of $\mathcal{C}(n)$ over \mathbb{F}_2 . Let $1_k \in C^k(\mathcal{C}(n), \mathbb{F}_2)$ denote the k -cocycle that sends all k -cells to 1.

The *standard sign assignment* $s \in C^1(\mathcal{C}(n), \mathbb{F}_2)$ (denoted s_0 in [LS, Definition 4.5]) is the following 1-cochain. If $u = (\epsilon_1, \dots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \dots, \epsilon_n)$ and $v = (\epsilon_1, \dots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \dots, \epsilon_n)$, then

$$s(\mathcal{C}_{u,v}) = (\epsilon_1 + \dots + \epsilon_{i-1}) \pmod{2} \in \mathbb{F}_2.$$

It is easy to see that $\delta s = 1_2$.

The *standard frame assignment* $f \in C^2(\mathcal{C}(n), \mathbb{F}_2)$ is the following 2-cochain. If $u = (\epsilon_1, \dots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \dots, \epsilon_{j-1}, 1, \epsilon_{j+1}, \dots, \epsilon_n)$ and $v = (\epsilon_1, \dots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \dots, \epsilon_{j-1}, 0, \epsilon_{j+1}, \dots, \epsilon_n)$, then

$$f(\mathcal{C}_{u,v}) = (\epsilon_1 + \dots + \epsilon_{i-1})(\epsilon_{i+1} + \dots + \epsilon_{j-1}) \pmod{2} \in \mathbb{F}_2.$$

Lemma 2.1. *For any $v \leq_3 u$,*

$$(\delta f)(\mathcal{C}_{u,v}) = \sum_{w \in \{w \mid v \leq_1 w \leq_2 u\}} s(\mathcal{C}_{w,v}).$$

Proof. Let $u = (\epsilon_1, \dots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \dots, \epsilon_{j-1}, 1, \epsilon_{j+1}, \dots, \epsilon_{k-1}, 1, \epsilon_{k+1}, \dots, \epsilon_n)$ and $v = (\epsilon_1, \dots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \dots, \epsilon_{j-1}, 0, \epsilon_{j+1}, \dots, \epsilon_{k-1}, 0, \epsilon_{k+1}, \dots, \epsilon_n)$. Then,

$$\begin{aligned} \sum_{w \in \{w \mid v \leq_1 w \leq_2 u\}} s(\mathcal{C}_{w,v}) &= (\epsilon_1 + \dots + \epsilon_{i-1}) + (\epsilon_1 + \dots + \epsilon_{j-1}) + (\epsilon_1 + \dots + \epsilon_{k-1}) \\ &= (\epsilon_1 + \dots + \epsilon_{i-1}) + (\epsilon_{j+1} + \dots + \epsilon_{k-1}). \end{aligned}$$

On the other hand,

$$\begin{aligned}
(\delta f)(\mathcal{C}_{u,v}) &= (\epsilon_1 + \cdots + \epsilon_{i-1})(\epsilon_{i+1} + \cdots + \epsilon_{j-1}) + (\epsilon_1 + \cdots + \epsilon_{i-1})(\epsilon_{i+1} + \cdots + \epsilon_{j-1}) \\
&\quad + (\epsilon_1 + \cdots + \epsilon_{i-1})(\epsilon_{i+1} + \cdots + \epsilon_{j-1} + 0 + \epsilon_{j+1} + \cdots + \epsilon_{k-1}) \\
&\quad + (\epsilon_1 + \cdots + \epsilon_{i-1})(\epsilon_{i+1} + \cdots + \epsilon_{j-1} + 1 + \epsilon_{j+1} + \cdots + \epsilon_{k-1}) \\
&\quad + (\epsilon_1 + \cdots + \epsilon_{i-1} + 0 + \epsilon_{i+1} + \cdots + \epsilon_{j-1})(\epsilon_{j+1} + \cdots + \epsilon_{k-1}) \\
&\quad + (\epsilon_1 + \cdots + \epsilon_{i-1} + 1 + \epsilon_{i+1} + \cdots + \epsilon_{j-1})(\epsilon_{j+1} + \cdots + \epsilon_{k-1}) \\
&= (\epsilon_1 + \cdots + \epsilon_{i-1}) + (\epsilon_{j+1} + \cdots + \epsilon_{k-1}),
\end{aligned}$$

thus completing the proof. \square

2.2. The Khovanov setup. In this subsection, we recall the definition of the Khovanov chain complex associated to an oriented link diagram L . Assume L has n crossings that have been ordered, and let n_- denote the number of negative crossings in L . In what follows, we will usually work over \mathbb{F}_2 , and we will always have a fixed n -crossing link diagram L in the background. Hence, we will typically drop both \mathbb{F}_2 and L from the notation, writing $KC = KC_{\mathbb{F}_2}(L)$ for the Khovanov complex of L with \mathbb{F}_2 -coefficients and $KC_{\mathbb{Z}}$ for the Khovanov complex of L with \mathbb{Z} -coefficients.

Given a vertex $u \in \{0, 1\}^n$, let $D(u)$ be the corresponding complete resolution of the link diagram L , where we take the 0 resolution at i^{th} crossing if $u_i = 0$, and the 1-resolution otherwise. We usually view $D(u)$ as a *resolution configuration* in the sense of [LS, Definition 2.1]; that is, we add arcs at the 0-resolutions to record the crossings. The set of circles (resp. arcs) that appear in $D(u)$ is denoted $Z(D(u))$ (resp. $A(D(u))$).

The *Khovanov generators* are of the form $\mathbf{x} = (D(u), x)$, where x is a labeling of the circles in $Z(D(u))$ by elements of $\{x_+, x_-\}$. Each Khovanov generator carries a *bigrading* $(\text{gr}_h, \text{gr}_q)$; gr_h is called the homological grading and gr_q is called the quantum grading. The bigrading is defined by:

$$\begin{aligned}
\text{gr}_h(D(u), x) &= -n_- + |u| \\
\text{gr}_q(D(u), x) &= n - 3n_- + |u| + \#\{Z \in Z(D(u)) \mid x(Z) = x_+\} \\
&\quad - \#\{Z \in Z(D(u)) \mid x(Z) = x_-\}.
\end{aligned}$$

The set of all Khovanov generators in bigrading (i, j) is denoted $KG^{i,j}$. There is an obvious map $\mathcal{F}: KG \rightarrow \{0, 1\}^n$ that sends $(D(u), x)$ to u . It is clear that if $\mathbf{x} \in KG^{i,j}$, then $|\mathcal{F}(\mathbf{x})| = n_- + i$.

The *Khovanov chain group* in bigrading (i, j) , $KC^{i,j}$, is the \mathbb{F}_2 vector space with basis $KG^{i,j}$; for $\mathbf{x} \in KG^{i,j}$, and $\mathbf{c} \in KC^{i,j}$, we say $\mathbf{x} \in \mathbf{c}$ if the coefficient of \mathbf{x} in \mathbf{c} is 1, and $\mathbf{x} \notin \mathbf{c}$ otherwise.

The *Khovanov differential* δ maps $KC^{i,j} \rightarrow KC^{i+1,j}$, and is defined as follows. If $\mathbf{y} = (D(v), y) \in KG^{i+1,j}$ and $\mathbf{x} = (D(u), x) \in KG^{i,j}$, then $\mathbf{x} \in \delta\mathbf{y}$ if the following hold:

- (1) $v \leq_1 u$, that is, $D(u)$ is obtained from $D(v)$ by performing an embedded 1-surgery along some arc $A_1 \in A(D(v))$. In particular, either,
 - (a) the endpoints of A_1 lie on the same circle, say $Z_1 \in D(v)$, which corresponds to two circles, say $Z_2, Z_3 \in D(u)$; or,
 - (b) The endpoints of A_1 lie on two different circles, say $Z_1, Z_2 \in D(v)$, which correspond to a single circle, say $Z_3 \in D(u)$.
- (2) In Case (1a), x and y induce the same labeling on $D(u) \setminus \{Z_2, Z_3\} = D(v) \setminus \{Z_1\}$; in Case (1b), x and y induce the same labeling on $D(u) \setminus \{Z_3\} = D(v) \setminus \{Z_1, Z_2\}$;
- (3) In Case (1a), either $y(Z_1) = x(Z_2) = x(Z_3) = x_-$ or $y(Z_1) = x_+$ and $\{x(Z_2), x(Z_3)\} = \{x_+, x_-\}$; in Case (1b), either $y(Z_1) = y(Z_2) = x(Z_3) = x_+$ or $\{y(Z_1), y(Z_2)\} = \{x_+, x_-\}$ and $x(Z_3) = x_-$.

It is clear that if $\mathbf{x} \in \delta \mathbf{y}$, then $\mathcal{F}(\mathbf{y}) \leq_1 \mathcal{F}(\mathbf{x})$. The *Khovanov homology* is the homology of (KC, δ) ; the Khovanov homology in bigrading (i, j) is denoted $Kh^{i,j}$. For a cycle $\mathbf{c} \in KC^{i,j}$, let $[\mathbf{c}] \in Kh^{i,j}$ denote the corresponding homology element.

2.3. A first look at the Khovanov space. The Khovanov chain complex is actually defined over \mathbb{Z} , and the \mathbb{F}_2 versions is its mod 2 reduction. The Khovanov chain group over \mathbb{Z} in bigrading (i, j) , $KC_{\mathbb{Z}}$, is the free \mathbb{Z} -module with basis $KG^{i,j}$. The differential $\delta_{\mathbb{Z}}: KC_{\mathbb{Z}}^{i,j} \rightarrow KC_{\mathbb{Z}}^{i+1,j}$ is defined by

$$(2.1) \quad \delta_{\mathbb{Z}} \mathbf{y} = \sum_{\mathbf{x} \in \delta \mathbf{y}} (-1)^{s(\mathcal{C}_{\mathcal{F}(\mathbf{x})}, \mathcal{F}(\mathbf{y}))} \mathbf{x}.$$

In [LS, Theorem 1], we construct Khovanov spectra \mathcal{X}_{Kh}^j satisfying $\tilde{H}^i(\mathcal{X}_{Kh}^j) = Kh_{\mathbb{Z}}^{i,j}$. Moreover, the spectrum $\mathcal{X}_{Kh} = \bigvee_j \mathcal{X}_{Kh}^j$ is defined as the suspension spectrum of a CW complex $|\mathcal{C}_K|$, formally desuspended a few times [LS, Definition 5.5] (this space is denoted $Y = \bigvee_j Y_j$ in Section 3). Furthermore, there is a bijection between the cells (except the basepoint) of $|\mathcal{C}_K|$ and the Khovanov generators in KG , which induces an isomorphism between $\tilde{C}^*(|\mathcal{C}_K|)$, the reduced cellular cochain complex, and $(KC_{\mathbb{Z}}, \delta_{\mathbb{Z}})$.

This allows us to associate homotopy invariants to Khovanov homology. Let \mathcal{A} be the (graded) Steenrod algebra over \mathbb{F}_2 , and let $\mathcal{A}(1)$ be the subalgebra generated by Sq^1 and Sq^2 . The Steenrod algebra \mathcal{A} acts on the Khovanov homology Kh , viewed as the (reduced) cohomology of the spectrum \mathcal{X}_{Kh} . The (stable) homotopy type of \mathcal{X}_{Kh} is a knot invariant, and therefore, the action of \mathcal{A} on Kh is a knot invariant as well.

2.4. The ladybug matching. Let $\mathbf{x} \in KG^{i+2,j}$ and $\mathbf{y} \in KG^{i,j}$ be Khovanov generators. Consider the set of Khovanov generators between \mathbf{x} and \mathbf{y} :

$$\mathcal{G}_{\mathbf{x}, \mathbf{y}} = \{\mathbf{z} \in KG^{i+1,j} \mid \mathbf{x} \in \delta \mathbf{z}, \mathbf{z} \in \delta \mathbf{y}\}.$$

Since δ is a differential, for all \mathbf{x}, \mathbf{y} , there are an even number of elements in $\mathcal{G}_{\mathbf{x}, \mathbf{y}}$. It is well-known that this even number is 0, 2 or 4. Indeed,

Lemma 2.2. [LS, Lemma 5.7] *Let $\mathbf{x} = (D(u), x)$ and $\mathbf{y} = (D(v), y)$. The set $\mathcal{G}_{\mathbf{x}, \mathbf{y}}$ has 4 elements if and only if the following hold.*

- (1) $v \leq_2 u$, that is, $D(u)$ is obtained from $D(v)$ by doing embedded 1-surgeries along two arcs, say $A_1, A_2 \in A(D(v))$.
- (2) The endpoints of A_1 and A_2 all lie on the same circle, say $Z_1 \in Z(D(v))$. Furthermore, their endpoints are linked on Z_1 , so Z_1 gives rise to a single circle, say Z_2 , in $Z(D(u))$.
- (3) x and y agree on $Z(D(u)) \setminus \{Z_2\} = Z(D(v)) \setminus \{Z_1\}$.
- (4) $y(Z_1) = x_+$ and $x(Z_2) = x_-$.

In the construction of the Khovanov space, we made a global choice. This choice furnishes us with a *ladybug matching* \mathfrak{l} , which is a collection $\{\mathfrak{l}_{\mathbf{x}, \mathbf{y}}\}$, for $\mathbf{x}, \mathbf{y} \in KG$ with $|\mathcal{F}(\mathbf{x})| = |\mathcal{F}(\mathbf{y})| + 2$, of fixed point free involutions $\mathfrak{l}_{\mathbf{x}, \mathbf{y}}: \mathcal{G}_{\mathbf{x}, \mathbf{y}} \rightarrow \mathcal{G}_{\mathbf{x}, \mathbf{y}}$. The ladybug matching is defined as follows.

Fix $\mathbf{x} = (D(u), x)$ and $\mathbf{y} = (D(v), y)$ in KG with $|u| = |v| + 2$; we will describe a fixed point free involution $\mathfrak{l}_{\mathbf{x}, \mathbf{y}}$ of $\mathcal{G}_{\mathbf{x}, \mathbf{y}}$. The only case of interest is when $\mathcal{G}_{\mathbf{x}, \mathbf{y}}$ has 4 elements; hence assume that we are in the case described in Lemma 2.2. Do an isotopy in S^2 so that $D(v)$ looks like Figure 2.1a. (In the figure, we have not shown the circles in $Z(D(v)) \setminus \{Z_1\}$ and the arcs in $A(D(v)) \setminus \{A_1, A_2\}$.) Figure 2.1b shows the four generators in $\mathcal{G}_{\mathbf{x}, \mathbf{y}}$ and the ladybug matching $\mathfrak{l}_{\mathbf{x}, \mathbf{y}}$. (Once again, we have not shown the extra circles and arcs.) It is easy to check (cf. [LS, Lemma 5.8]) that this matching is well-defined, i.e., it is independent of the choice of isotopy and the numbering of the two arcs in $A(D(v)) \setminus A(D(u))$ as $\{A_1, A_2\}$.

Lemma 2.3. *Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be Khovanov generators with $\mathbf{z} \in \mathcal{G}_{\mathbf{x}, \mathbf{y}}$. Let $\mathbf{z}' = \mathfrak{l}_{\mathbf{x}, \mathbf{y}}(\mathbf{z})$. Then*

$$s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{z})}) + s(\mathcal{C}_{\mathcal{F}(\mathbf{z}), \mathcal{F}(\mathbf{y})}) + s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{z}')}) + s(\mathcal{C}_{\mathcal{F}(\mathbf{z}'), \mathcal{F}(\mathbf{y})}) = 1.$$

Proof. Let $u = \mathcal{F}(\mathbf{x})$, $v = \mathcal{F}(\mathbf{y})$, $w = \mathcal{F}(\mathbf{z})$ and $w' = \mathcal{F}(\mathbf{z}')$. We have $v \leq_1 w, w' \leq_1 u$.

It follows from the definition of ladybug matching (Figure 2.1b) that $w \neq w'$. Therefore, u, v, w, w' are precisely the four vertices that appear in the 2-cell $\mathcal{C}_{u, v}$. Since $\delta s = 1_2$,

$$\delta s(\mathcal{C}_{u, v}) = s(\mathcal{C}_{u, w}) + s(\mathcal{C}_{w, v}) + s(\mathcal{C}_{u, w'}) + s(\mathcal{C}_{w', v}) = 1. \quad \square$$

2.5. The operation Sq^1 . Let $\mathbf{c} \in KC^{i, j}$ be a cycle in the Khovanov chain complex. For $\mathbf{x} \in KG^{i+1, j}$, let $\mathcal{G}_{\mathbf{c}}(\mathbf{x}) = \{\mathbf{y} \in KG^{i, j} \mid \mathbf{x} \in \delta \mathbf{y}, \mathbf{y} \in \mathbf{c}\}$.

Definition 2.4. A *boundary matching* \mathbf{m} for \mathbf{c} is a collection of pairs $(\mathbf{b}_{\mathbf{x}}, \mathbf{s}_{\mathbf{x}})$, one for each $\mathbf{x} \in KG^{i+1, j}$, where:

- $\mathbf{b}_{\mathbf{x}}$ is a fixed point free involution of $\mathcal{G}_{\mathbf{c}}(\mathbf{x})$, and
- $\mathbf{s}_{\mathbf{x}}$ is a map $\mathcal{G}_{\mathbf{c}}(\mathbf{x}) \rightarrow \mathbb{F}_2$, such that for all $\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x})$,

$$\{\mathbf{s}_{\mathbf{x}}(\mathbf{y}), \mathbf{s}_{\mathbf{x}}(\mathbf{b}_{\mathbf{x}}(\mathbf{y}))\} = \begin{cases} \{0, 1\} & \text{if } s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{y})}) = s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{b}_{\mathbf{x}}(\mathbf{y}))}) \\ \{0\} & \text{otherwise.} \end{cases}$$

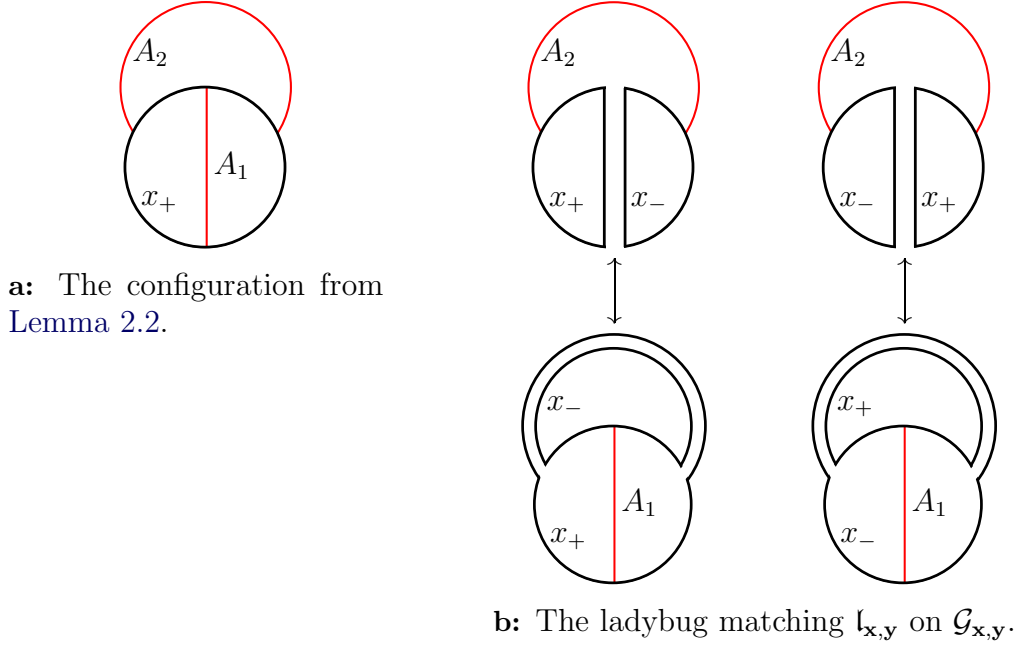


FIGURE 2.1. **The ladybug matching.** We have shown the case when $\mathcal{G}_{\mathbf{x}, \mathbf{y}}$ has 4 elements.

Since \mathbf{c} is a cycle, for any \mathbf{x} there are an even number of elements in $\mathcal{G}_{\mathbf{c}}(\mathbf{x})$. Hence, there exists a boundary matching \mathbf{m} for \mathbf{c} .

Definition 2.5. Let $\mathbf{c} \in KC^{i,j}$ be a cycle. For any boundary matching $\mathbf{m} = \{(\mathfrak{b}_{\mathbf{x}}, \mathfrak{s}_{\mathbf{x}})\}$ for \mathbf{c} , define the chain $\text{sq}_{\mathbf{m}}^1(\mathbf{c}) \in KC^{i+1,j}$ as

$$(2.2) \quad \text{sq}_{\mathbf{m}}^1(\mathbf{c}) = \sum_{\mathbf{x} \in KG^{i+1,j}} \left(\sum_{\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x})} \mathfrak{s}_{\mathbf{x}}(\mathbf{y}) \right) \mathbf{x}.$$

Proposition 2.6. For any cycle $\mathbf{c} \in KC^{i,j}$ and any boundary matching \mathbf{m} for \mathbf{c} , $\text{sq}_{\mathbf{m}}^1(\mathbf{c})$ is a cycle. Furthermore,

$$[\text{sq}_{\mathbf{m}}^1(\mathbf{c})] = \text{Sq}^1([\mathbf{c}]).$$

Proof. The first Steenrod square Sq^1 is the Bockstein associated to the short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Since the differential in $KC_{\mathbb{Z}}$ is given by Equation (2.1), a chain representative for $\text{Sq}^1([c])$ is the following:

$$\mathbf{b} = \sum_{\mathbf{x} \in KG^{i+1,j}} \left(\frac{\#\{\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x}) \mid s(\mathcal{C}_{\mathcal{F}(\mathbf{x}),\mathcal{F}(\mathbf{y})}) = 0\} - \#\{\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x}) \mid s(\mathcal{C}_{\mathcal{F}(\mathbf{x}),\mathcal{F}(\mathbf{y})}) = 1\}}{2} \right) \mathbf{x}.$$

It is easy to see that

$$\sum_{\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x})} \mathbf{s}_{\mathbf{x}}(\mathbf{y}) = \left(\frac{\#\{\mathbf{y} \mid s(\mathcal{C}_{\mathcal{F}(\mathbf{x}),\mathcal{F}(\mathbf{y})}) = 0\} - \#\{\mathbf{y} \mid s(\mathcal{C}_{\mathcal{F}(\mathbf{x}),\mathcal{F}(\mathbf{y})}) = 1\}}{2} \right) \pmod{2},$$

and hence $\mathbf{b} = \text{sq}_{\mathbf{m}}^1(\mathbf{c})$. \square

2.6. The operation Sq^2 . Let $\mathbf{c} \in KC^{i,j}$ be a cycle. Choose a boundary matching $\mathbf{m} = \{(\mathbf{b}_{\mathbf{z}}, \mathbf{s}_{\mathbf{z}})\}$ for \mathbf{c} . For $\mathbf{x} \in KG^{i+2,j}$, define

$$\mathcal{G}_{\mathbf{c}}(\mathbf{x}) = \{(\mathbf{z}, \mathbf{y}) \in KG^{i+1,j} \times KG^{i,j} \mid \mathbf{x} \in \delta \mathbf{z}, \mathbf{z} \in \delta \mathbf{y}, \mathbf{y} \in \mathbf{c}\}.$$

Consider the edge-labeled graph $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$, whose vertices are the elements of $\mathcal{G}_{\mathbf{c}}(\mathbf{x})$ and whose edges are the following.

- (e-1) There is an unoriented edge between (\mathbf{z}, \mathbf{y}) and $(\mathbf{z}', \mathbf{y})$, if the ladybug matching $\mathbf{l}_{\mathbf{x},\mathbf{y}}$ matches \mathbf{z} and \mathbf{z}' . This edge is labeled by $f(\mathcal{C}_{\mathcal{F}(\mathbf{x}),\mathcal{F}(\mathbf{y})}) \in \mathbb{F}_2$, where f denotes the standard frame assignment (Subsection 2.1).
- (e-2) There is an edge between (\mathbf{z}, \mathbf{y}) and $(\mathbf{z}, \mathbf{y}')$ if the matching $\mathbf{b}_{\mathbf{z}}$ matches \mathbf{y} with \mathbf{y}' . This edge is labeled by 0. Furthermore, if $\mathbf{s}_{\mathbf{z}}(\mathbf{y}) = 0$ and $\mathbf{s}_{\mathbf{z}}(\mathbf{y}') = 1$, then this edge is oriented from (\mathbf{z}, \mathbf{y}) to $(\mathbf{z}, \mathbf{y}')$; if $\mathbf{s}_{\mathbf{z}}(\mathbf{y}) = 1$ and $\mathbf{s}_{\mathbf{z}}(\mathbf{y}') = 0$, then this edge is oriented from $(\mathbf{z}, \mathbf{y}')$ to (\mathbf{z}, \mathbf{y}) ; and if $\mathbf{s}_{\mathbf{z}}(\mathbf{y}) = \mathbf{s}_{\mathbf{z}}(\mathbf{y}')$, then the edge is unoriented.

Let $f(\mathfrak{G}_{\mathbf{c}}(\mathbf{x})) \in \mathbb{F}_2$ be the sum of all the edge-labels in the graph.

Lemma 2.7. *Each component of $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ is an even cycle. Furthermore, in each component, the number of oriented edges is even.*

Proof. Each vertex (\mathbf{z}, \mathbf{y}) of $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ belongs to exactly two edges: the Type (e-1) edge joining (\mathbf{z}, \mathbf{y}) and $(\mathbf{l}_{\mathbf{x},\mathbf{y}}(\mathbf{z}), \mathbf{y})$; and the Type (e-2) edge joining (\mathbf{z}, \mathbf{y}) and $(\mathbf{z}, \mathbf{b}_{\mathbf{z}}(\mathbf{y}))$. This implies that each component of $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ is an even cycle.

In order to prove the second part, vertex-label the graph as follows: To a vertex (\mathbf{z}, \mathbf{y}) , assign the number $s(\mathcal{C}_{\mathcal{F}(\mathbf{x}),\mathcal{F}(\mathbf{z})}) + s(\mathcal{C}_{\mathcal{F}(\mathbf{z}),\mathcal{F}(\mathbf{y})}) \in \mathbb{F}_2$. Lemma 2.3 implies that the Type (e-1) edges join vertices carrying opposite labels; and among the Type (e-2) edges, it is clear from the definition of boundary matching (Subsection 2.5) that the oriented edges join vertices carrying the same label, and the unoriented edges join vertices carrying opposite labels. Therefore, each cycle must contain an even number of unoriented edges; since there are an even number of vertices in each cycle, we are done. \square

This observation allows us to associate a number $g(\mathfrak{G}_c(\mathbf{x})) \in \mathbb{F}_2$ to the graph: Partition the oriented edges of $\mathfrak{G}_c(\mathbf{x})$ into two sets, such that if two edges from the same cycle are in the same set, they are oriented in the same direction; then $g(\mathfrak{G}_c(\mathbf{x}))$ is the number modulo 2 of the elements in either set.

Definition 2.8. Let $\mathbf{c} \in KC^{i,j}$ be a cycle. For any boundary matching \mathbf{m} for \mathbf{c} , define the chain $\text{sq}_{\mathbf{m}}^2(\mathbf{c}) \in KC^{i+1,j}$ as

$$(2.3) \quad \text{sq}_{\mathbf{m}}^2(\mathbf{c}) = \sum_{\mathbf{x} \in KG^{i+2,j}} \left(\#|\mathfrak{G}_c(\mathbf{x})| + f(\mathfrak{G}_c(\mathbf{x})) + g(\mathfrak{G}_c(\mathbf{x})) \right) \mathbf{x}.$$

We devote [Section 3](#) to proving the following.

Theorem 2. For any cycle $\mathbf{c} \in KC^{i,j}$ and any boundary matching \mathbf{m} for \mathbf{c} , $\text{sq}_{\mathbf{m}}^2(\mathbf{c})$ is a cycle. Furthermore,

$$[\text{sq}_{\mathbf{m}}^2(\mathbf{c})] = \text{Sq}^2([\mathbf{c}]).$$

Corollary 2.9. The operations $\text{sq}_{\mathbf{m}}^1$ and $\text{sq}_{\mathbf{m}}^2$ induce well-defined maps

$$\text{Sq}^1: Kh^{i,j} \rightarrow Kh^{i+1,j} \quad \text{and} \quad \text{Sq}^2: Kh^{i,j} \rightarrow Kh^{i+2,j}$$

that are independent of the choices of boundary matchings \mathbf{m} . Furthermore, these maps are link invariants, in the following sense: given any two diagrams L and L' representing the same link, there are isomorphisms $\phi_{i,j}: Kh^{i,j}(L) \rightarrow Kh^{i,j}(L')$ making the following diagrams commute:

$$\begin{array}{ccc} Kh^{i+1,j}(L) & \xrightarrow{\phi_{i+1,j}} & Kh^{i+1,j}(L') \\ \uparrow \text{Sq}^1 & & \uparrow \text{Sq}^1 \\ Kh^{i,j}(L) & \xrightarrow{\phi_{i,j}} & Kh^{i,j}(L') \end{array} \quad \begin{array}{ccc} Kh^{i+2,j}(L) & \xrightarrow{\phi_{i+2,j}} & Kh^{i+2,j}(L') \\ \uparrow \text{Sq}^2 & & \uparrow \text{Sq}^2 \\ Kh^{i,j}(L) & \xrightarrow{\phi_{i,j}} & Kh^{i,j}(L'). \end{array}$$

Proof. This is immediate from [Proposition 2.6](#), [Theorem 2](#), and invariance of the Khovanov spectrum [[LS](#), Theorem 1]. \square

3. WHERE THE ANSWER COMES FROM

This section is devoted to proving [Theorem 2](#). The operation Sq^2 on a CW complex Y is determined by the sub-quotients $Y^{(m+2)}/Y^{(m-1)}$. (In [Subsection 3.1](#) we review an explicit description in these terms, due to Steenrod.) The space $\mathcal{X}_{Kh}(L)$ is a formal de-suspension of a CW complex $Y = Y(L)$. So, most of the work is in understanding combinatorially how the m -, $(m+1)$ - and $(m+2)$ -cells of $Y(L)$ are glued together.

The description of $Y(L)$ from [[LS](#)] is in terms of a Pontrjagin-Thom type construction. To understand just $Y^{(m+2)}/Y^{(m-1)}$ involves studying certain framed points in \mathbb{R}^m and framed paths in $\mathbb{R} \times \mathbb{R}^m$. We will draw these framings from a particular set of choices,

described in Subsection 3.2. Subsection 3.3 explains exactly how we assign framings from this set, and shows that these framings are consistent with the construction in [LS]. Finally, Subsection 3.4 discusses how to go from these choices to $Y^{(m+2)}/Y^{(m-1)}$, and why the resulting operation Sq^2 agrees with the operation from Definition 2.8.

3.1. Sq^2 for a CW complex. We start by recalling a definition of Sq^2 . The discussion in this section is heavily inspired by [Ste72, Section 12].

Let $K_m = K(\mathbb{Z}/2, m)$ denote the m^{th} Eilenberg-MacLane space for the group $\mathbb{Z}/2$, so $\pi_m(K_m) = \mathbb{Z}/2$ and $\pi_i(K_m) = 0$ for $i \neq m$. Assume that m is sufficiently large, say $m \geq 3$. We start by discussing a CW structure for K_m . Since $\pi_i(K_m) = 0$ for $i < m$, we can choose the m -skeleton $K_m^{(m)}$ to be a single m -cell e^m with the entire boundary ∂e^m attached to the basepoint. To arrange that $\pi_m(K_m) = \mathbb{Z}/2$ it suffices to attach a single $(m+1)$ -cell via a degree 2 map $\partial e^{m+1} \rightarrow K_m^{(m)} = S^m$.

We show that the resulting $(m+1)$ -skeleton $K_m^{(m+1)}$ has $\pi_{m+1}(K_m^{(m+1)}) \cong \mathbb{Z}/2$. From the long exact sequence for the pair (K_m^{m+1}, S^m) ,

$$\pi_{m+2}(K_m^{(m+1)}, S^m) \rightarrow \pi_{m+1}(S^m) \rightarrow \pi_{m+1}(K_m^{(m+1)}) \rightarrow \pi_{m+1}(K_m^{(m+1)}, S^m) \rightarrow \pi_m(S^m).$$

By excision (since m is large), $\pi_{m+1}(K_m^{(m+1)}, S^m) \cong \pi_{m+1}(K_m^{(m+1)}/S^m) = \pi_{m+1}(S^{m+1}) = \mathbb{Z}$ and $\pi_{m+2}(K_m^{(m+1)}, S^m) \cong \pi_{m+2}(S^{m+1}) = \mathbb{Z}/2$. The maps $\pi_{i+1}(K_m^{(m+1)}, S^m) = \pi_{i+1}(S^{m+1}) \rightarrow \pi_i(S^m)$ are twice the Freudenthal isomorphisms. So, this sequence becomes

$$\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2 \rightarrow \pi_{m+1}(K_m^{(m+1)}) \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z}.$$

Thus, $\pi_{m+1}(K_m^{(m+1)}) \cong \mathbb{Z}/2$, represented by the Hopf map $S^{m+1} \rightarrow S^m = K_m^{(m)} \hookrightarrow K_m^{(m+1)}$.

Let $K_m^{(m+2)}$ be the result of attaching an $(m+2)$ -cell e^{m+2} to kill this $\mathbb{Z}/2$. This attaching map has degree 0 as a map to the $(m+1)$ -cell e^{m+1} in $K_m^{(m+1)}$, so the $(m+2)$ -skeleton of K_m has cohomology:

$$\begin{aligned} H^m(K_m^{(m+2)}; \mathbb{Z}) &= \mathbb{F}_2 & H^{m+1}(K_m^{(m+2)}; \mathbb{Z}) &= 0 & H^{m+2}(K_m^{(m+2)}; \mathbb{Z}) &= \mathbb{Z} \\ H^m(K_m^{(m+2)}; \mathbb{F}_2) &= \mathbb{F}_2 & H^{m+1}(K_m^{(m+2)}; \mathbb{F}_2) &= \mathbb{F}_2 & H^{m+2}(K_m^{(m+2)}; \mathbb{F}_2) &= \mathbb{F}_2. \end{aligned}$$

Hence, there are fundamental cohomology classes $\tilde{\iota} \in H^m(K_m; \mathbb{Z})$ and $\iota \in H^m(K_m; \mathbb{F}_2)$, so that ι is the mod-2 reduction of $\tilde{\iota}$; and elements $\widetilde{\text{Sq}}^2(\tilde{\iota}) \in H^{m+2}(K_m^{(m+2)}; \mathbb{Z})$ and $\text{Sq}^2(\iota) \in H^{m+2}(K_m^{(m+2)}; \mathbb{F}_2)$, so that $\text{Sq}^2(\iota)$ is the mod-2 reduction of $\widetilde{\text{Sq}}^2(\tilde{\iota})$. It turns out that the element $\widetilde{\text{Sq}}^2(\tilde{\iota})$ does not survive to $H^{m+2}(K_m; \mathbb{Z})$, but $\text{Sq}^2(\iota)$ does survive to $H^{m+2}(K_m; \mathbb{F}_2)$.

Now, consider a CW complex Y and a cohomology class $c \in H^m(Y; \mathbb{F}_2)$. The element c is classified by a map $\mathbf{c}: Y \rightarrow K_m$, so that $\mathbf{c}^* \iota = c$. We can arrange that the map \mathbf{c} is cellular. So, we have an element $\text{Sq}^2(c) = \mathbf{c}^* \text{Sq}^2(\iota) \in H^{m+2}(Y; \mathbb{F}_2)$.

The element $\text{Sq}^2(c)$ is determined by its restriction to $H^{m+2}(Y^{(m+2)}; \mathbb{F}_2)$. So, to compute $\text{Sq}^2(c)$ it suffices to give a cellular map $Y^{(m+2)} \rightarrow K_m^{(m+2)}$ so that ι pulls back to c . Then,

$\text{Sq}^2(c)$ is the cochain which sends an $(m+2)$ -cell f^{m+2} of Y to the degree of the map $f^{m+2}/\partial f^{m+2} \rightarrow e^{m+2}/\partial e^{m+2}$. Equivalently, $\text{Sq}^2(c)$ sends f^{m+2} to the element $\mathbf{c}|_{\partial f^{m+2}} \in \pi_{m+1}(K_m^{(m+1)}) = \pi_{m+1}(S^m) = \mathbb{Z}/2$. (In other words, $\text{Sq}^2(c)$ is the obstruction to homotoping \mathbf{c} so that it sends the $(m+2)$ -skeleton of Y to the $(m+1)$ -skeleton of K_m .) Since $K_m^{(m+2)}$ has no cells of dimension between 0 and m , the map $Y^{(m+2)} \rightarrow K_m^{(m+2)}$ factors through $Y^{(m+2)}/Y^{(m-1)}$.

To understand the operation Sq^2 on Khovanov homology induced by the Khovanov homotopy type Y , it remains to explicitly give the map \mathbf{c} on $Y^{(m+2)}/Y^{(m-1)}$. This will be done in [Subsection 3.4](#), after we develop tools to understand the attaching maps for the $(m+2)$ -cells.

3.2. Frames in \mathbb{R}^3 . As discussed in [Subsection 3.4](#), the sub-quotients $Y^{(m+2)}/Y^{(m-1)}$ of the Khovanov space $Y = Y(L)$ are defined in terms of framed points in $\{0\} \times \mathbb{R}^m$ and framed paths in $\mathbb{R} \times \mathbb{R}^m$ connecting these points.

A framing of a path $\gamma: [0, 1] \rightarrow \mathbb{R}^{m+1}$ is a tuple $(v_1(t), \dots, v_m(t)) \in (\mathbb{R}^{m+1})^m$ of orthonormal vector fields along γ , normal to γ . A collection of m orthonormal vectors v_1, \dots, v_m in \mathbb{R}^{m+1} specifies a matrix in $SO(m+1)$, whose last m columns are v_1, \dots, v_m and whose first column is the cross product of v_1, \dots, v_m .

Now, suppose that $p, q \in \{0\} \times \mathbb{R}^m$ and that we are given trivializations φ_p, φ_q of the normal bundles in $\{0\} \times \mathbb{R}^m$ to p, q (i.e., framings of p and q). On the one hand, we can consider the set of isotopy classes of framed paths from (p, φ_p) to (q, φ_q) . On the other hand, we can consider the homotopy classes of paths in $SO(m+1)$ from φ_p to φ_q . There is an obvious map from isotopy classes of framed paths in \mathbb{R}^{m+1} to homotopy classes of paths in $SO(m+1)$, by considering only the framing. This map is a surjection if $m \geq 2$ and a bijection if $m \geq 3$. In the case that $m \geq 3$, both sets have two elements.

The upshot is that if we want to specify an isotopy classes of framed paths with given endpoints, and $m \geq 3$, then it suffices to specify a homotopy class of paths in $SO(m+1)$.

The framings on both the endpoints and the paths relevant to constructing $Y^{(m+2)}/Y^{(m-1)}$ will have a special form: they will be stabilizations of the $m = 2$ case. Specifically, we will write $m = m_1 + m_2$ with $m_i \geq 1$. Let $[\bar{e}, e_{11}, \dots, e_{1m_1}, e_{21}, \dots, e_{2m_2}]$ denote the standard basis for $\mathbb{R} \times \mathbb{R}^m$. Then all of the points will have framings of the form

$$(0, v_1, e_{12}, \dots, e_{1m_1}, v_2, e_{22}, \dots, e_{2m_2}) \in \{0\} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2},$$

for some v_1, v_2 . So, to describe isotopy classes of framed paths connecting these points it suffices to describe paths of the form

$$(v_0(t), v_1(t), e_{12}, \dots, e_{1m_1}, v_2(t), e_{22}, \dots, e_{2m_2}) \in \mathbb{R} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}.$$

Therefore, to describe such paths, it suffices to work in \mathbb{R}^3 (i.e., the case $m = 2$). Denote the standard basis for \mathbb{R}^3 by $[\bar{e}, e_1, e_2]$. We will work with the four distinguished frames in $\{0\} \times \mathbb{R}^2$, $(0, e_1, e_2), (0, -e_1, e_2), (0, e_1, -e_2), (0, -e_1, -e_2)$, which we denote by the symbols



FIGURE 3.1. **Null-homotopy of the loop $\overline{++} \cdot \overline{+-} \cdot \overline{-+}$ in $SO(3)$.** Viewing the arm as 2-dimensional, spanned by the tangent vector to the radius and the vector from the radius to the ulna, it traces out an extension of the map $S^1 \rightarrow SO(3)$ to a map $\mathbb{D}^2 \rightarrow SO(3)$.

$\overline{++}, \overline{+-}, \overline{-+}, \overline{--}$, respectively. By a *coherent system of paths* joining $\overline{++}, \overline{+-}, \overline{-+}, \overline{--}$ we mean a choice of a path $\overline{\varphi_1 \varphi_2}$ in $SO(3)$ from φ_1 to φ_2 for each pair of frames $\varphi_1, \varphi_2 \in \{\overline{++}, \overline{+-}, \overline{-+}, \overline{--}\}$, satisfying the following cocycle conditions:

- (1) For all $\varphi \in \{\overline{++}, \overline{+-}, \overline{-+}, \overline{--}\}$, the loop $\overline{\varphi \varphi}$ is nullhomotopic; and
- (2) For all $\varphi_1, \varphi_2, \varphi_3 \in \{\overline{++}, \overline{+-}, \overline{-+}, \overline{--}\}$, the path $\overline{\varphi_1 \varphi_2} \cdot \overline{\varphi_2 \varphi_3}$ is homotopic (relative endpoints) to the path $\overline{\varphi_1 \varphi_3}$.

We make a particular choice of a coherent system of paths, as follows:

- $\overline{++}, \overline{+-}, \overline{-+}, \overline{--}$: Rotate 180° around the e_2 -axis, such that the first vector equals \bar{e} halfway through.
- $\overline{+-}, \overline{--}$: Rotate 180° around the e_1 -axis, such that the second vector equals \bar{e} halfway through.
- $\overline{-+}, \overline{--}$: Rotate 180° around the e_1 -axis, such that the second vector equals $-\bar{e}$ halfway through.
- $\overline{+-}, \overline{-+}, \overline{--}, \overline{++}$: Rotate 180° around the \bar{e} -axis, such that the second vector equals $-e_1$ halfway through.

Lemma 3.1. *The above choice describes a coherent system of paths.*

Proof. We only need to check that each of the loops $\overline{++} \cdot \overline{+-} \cdot \overline{-+} \cdot \overline{--} \cdot \overline{++}$, $\overline{+-} \cdot \overline{--} \cdot \overline{-+} \cdot \overline{++} \cdot \overline{+-}$, and $\overline{++} \cdot \overline{+-} \cdot \overline{-+} \cdot \overline{--} \cdot \overline{++}$ are null-homotopic. This is best checked with hand motions, as we have illustrated for the first loop in Figure 3.1. \square

Extending this slightly:

Definition 3.2. Fix m_1, m_2 , and let $m = m_1 + m_2$. By the four *standard frames* for $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ we mean the frames

$$(0, \pm e_{11}, e_{12}, \dots, e_{1m_1}, \pm e_{21}, e_{22}, \dots, e_{2m_2}) \in \{0\} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}.$$

Up to homotopy, there are exactly two paths between any pair of frames. By the *standard frame paths* in $\mathbb{R} \times \mathbb{R}^m$ we mean the one-parameter families of frames obtained by extending the coherent system of paths for $SO(3)$ specified above by the identity on $\mathbb{R}^{m-2} = \mathbb{R}^{m_1-1} \times \mathbb{R}^{m_2-1}$. Abusing terminology, we will sometimes say that any frame path homotopic (relative endpoints) to a standard frame path is itself a standard frame path. By a *non-standard frame*

path we mean a frame path which is not homotopic (relative endpoints) to one of the standard frame paths.

Define

$$(3.1) \quad \mathbf{r}: \mathbb{R}^m \rightarrow \mathbb{R}^m \quad \mathbf{r}(x_1, \dots, x_m) = (-x_1, x_2, \dots, x_m)$$

$$(3.2) \quad \mathbf{s}: \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}^m \quad \mathbf{s}(x_1, \dots, x_m) = (x_1, \dots, x_{m_1}, -x_{m_1+1}, x_{m_1+2}, \dots, x_m).$$

Lemma 3.3. *Suppose φ_1, φ_2 are oppositely-oriented standard frames. Then $\mathbf{r}(\overline{\varphi_1 \varphi_2})$ is the non-standard frame path between $\mathbf{r}(\varphi_1)$ and $\mathbf{r}(\varphi_2)$. That is, \mathbf{r} takes standard frame paths between oppositely-oriented frames to non-standard frame paths.*

The map \mathbf{s} satisfies

$$\begin{array}{c} \overline{+-} \\ ++ \end{array} \xleftrightarrow{\mathbf{s}} \begin{array}{c} \overline{-+} \\ ++ \end{array} \quad \begin{array}{c} \overline{+-} \\ -- \end{array} \xleftrightarrow{\mathbf{s}} \begin{array}{c} \overline{-+} \\ -- \end{array}.$$

In other words, \mathbf{s} takes the standard frame path $\overline{\begin{smallmatrix} + & - \\ * & * \end{smallmatrix}}$ to the standard frame path $\overline{\begin{smallmatrix} - & + \\ * & * \end{smallmatrix}}$ for either $* \in \{+, -\}$.

Proof. This is a straightforward verification from the definitions. \square

3.3. The framed cube flow category. In this subsection, we describe certain aspects of the *flow category* $\mathcal{C}_C(n)$ associated to the cube $[0, 1]^n$. For a more complete account of the story, see [LS, Section 4]. The features of $\mathcal{C}_C(n)$ in which we are interested are the following:

- (F-1) To a pair of vertices $u, v \in \{0, 1\}^n$ with $v \leq_k u$, $\mathcal{C}_C(n)$ associates a $(k-1)$ -dimensional manifold with corners⁽ⁱ⁾ called the *moduli space* $\mathcal{M}_{\mathcal{C}_C(n)}(u, v)$. We drop the subscript if it is clear from the context.
- (F-2) For vertices $v < w < u$ in $\{0, 1\}^n$, $\mathcal{M}(w, v) \times \mathcal{M}(u, w)$ is identified with a subspace of $\partial \mathcal{M}(u, v)$.
- (F-3) Fix vertices $v \leq_k u$ in $\{0, 1\}^n$; let $\bar{0}, \bar{1} \in \{0, 1\}^k$ be the minimum and the maximum vertex, respectively. Then $\mathcal{M}_{\mathcal{C}_C(n)}(u, v)$ can be identified with $\mathcal{M}_{\mathcal{C}_C(k)}(\bar{1}, \bar{0})$.
- (F-4) $\mathcal{M}_{\mathcal{C}_C(n)}(\bar{1}, \bar{0})$ is a point, an interval and a hexagon, for $n = 1, 2, 3$, respectively, cf. Figure 3.2.

The cube flow category is also *framed*. In order to define framings, one needs to embed the moduli spaces into Euclidean spaces; one does so by *neat embeddings* (see [Lau00, Definition 2.1.4] or [LS, Definition 3.9]). Fix d sufficiently large; for each $v \leq_k u$, $\mathcal{M}(u, v)$ is neatly embedded in $\mathbb{R}_+^{k-1} \times \mathbb{R}^{kd}$. We require these embeddings to be coherent in the sense that for each $v \leq_k w \leq_l u$, $\mathcal{M}(w, v) \times \mathcal{M}(u, w) \subset \partial \mathcal{M}(u, v)$ is embedded by the product embedding into $\mathbb{R}_+^{k-1} \times \mathbb{R}^{kd} \times \mathbb{R}_+^{l-1} \times \mathbb{R}^{ld} = \mathbb{R}_+^{k-1} \times \{0\} \times \mathbb{R}_+^{l-1} \times \mathbb{R}^{kd+ld} \subset \partial(\mathbb{R}_+^{k+l-1} \times \mathbb{R}^{(k+l)d})$. The normal bundle to each of these moduli spaces is framed. We require these framings to be coherent in the sense that the product framing on $\mathcal{M}(w, v) \times \mathcal{M}(u, w)$ agrees with the framing induced from $\mathcal{M}(u, v)$.

⁽ⁱ⁾It is also a $\langle k-1 \rangle$ -manifold in the sense of [Lau00].

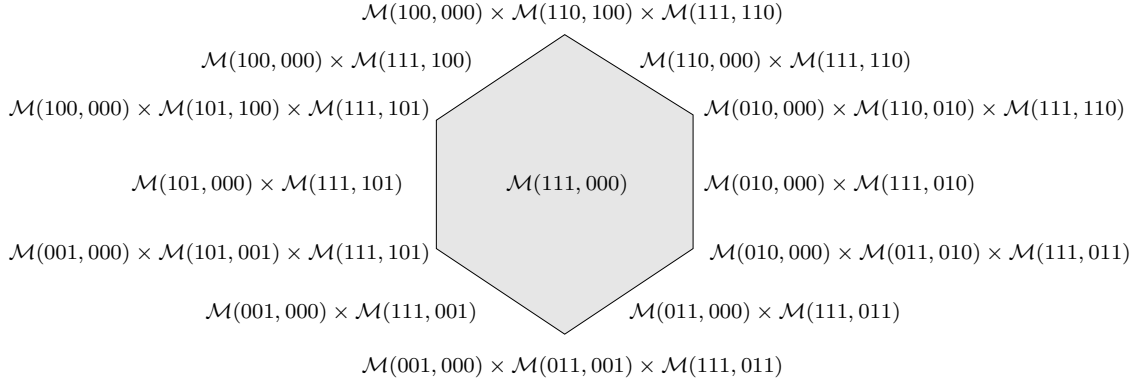


FIGURE 3.2. **The hexagon $\mathcal{M}_{\mathcal{C}(3)}(111, 000)$.** Each face corresponds to a product of lower-dimensional moduli spaces, as indicated.

The framed cube flow category $\mathcal{C}_C(n)$ is needed in the construction of the Khovanov homotopy type. The cube flow category can be framed in multiple ways. However, all such framings lead to the same Khovanov homotopy type [LS, Proposition 6.1]; hence it is enough to consider a specific framing. Consider the following partial framing.

Definition 3.4. Let $s \in C^1(\mathcal{C}(n), \mathbb{F}_2)$ and $f \in C^2(\mathcal{C}(n), \mathbb{F}_2)$ be the standard sign assignment and the standard frame assignment from Subsection 2.1. Fix d sufficiently large.

- Consider $v \leq_1 u$ in $\{0, 1\}^n$. Embed the point $\mathcal{M}(u, v)$ in \mathbb{R}^d ; let $[e_1, \dots, e_d]$ be the standard basis in \mathbb{R}^d . For framing the point $\mathcal{M}(u, v)$, choose the frame (e_1, e_2, \dots, e_d) if $s(\mathcal{C}_{u,v}) = 0$, and choose the frame $(-e_1, e_2, \dots, e_d)$ if $s(\mathcal{C}_{u,v}) = 1$.
- Consider $v \leq_2 u$ in $\{0, 1\}^n$; let w_1 and w_2 be the two other vertices in $\mathcal{C}_{u,v}$. Choose a proper embedding of the interval $\mathcal{M}(u, v)$ in $\mathbb{R}_+ \times \mathbb{R}^{2d}$; let $[\bar{e}, e_{11}, \dots, e_{1d}, e_{21}, \dots, e_{2d}]$ be the standard basis for $\mathbb{R} \times \mathbb{R}^{2d}$. The two endpoints $\mathcal{M}(w_i, v) \times \mathcal{M}(u, w_i)$ of the interval $\mathcal{M}(u, v)$ are already framed in $\{0\} \times \mathbb{R}^{2d}$ by the product framings, say φ_i . Since s is a sign assignment, the framings of the two endpoints, φ_1 and φ_2 , are opposite, and hence can be extended to a framing on the interval. Any such extension can be treated as a path joining φ_1 and φ_2 in $SO(2d+1)$, cf. Subsection 3.2. If $f(\mathcal{C}_{u,v}) = 0$, choose an extension so that the path is a standard frame path; if $f(\mathcal{C}_{u,v}) = 1$, choose an extension so that the path is a non-standard frame path.

As we will see in the Subsection 3.4, in order to study the Sq^2 action, one only needs to understand the framings of the 0-dimensional and the 1-dimensional moduli spaces. Therefore, the information encoded in Definition 3.4 is all we need in order to study the Sq^2 action. However, before we proceed onto the next subsection, we need to check the following.

Lemma 3.5. *The partial framing from Definition 3.4 can be extended to a framing of the entire cube flow category $\mathcal{C}_C(n)$.*

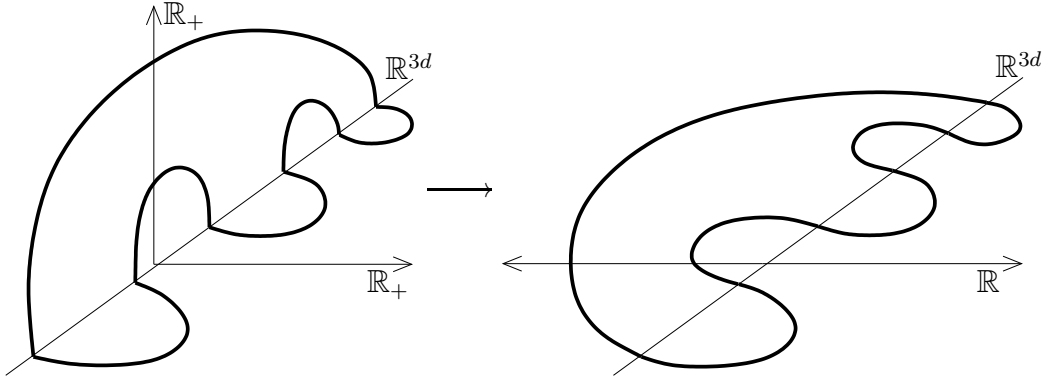


FIGURE 3.3. **The embedding of $\partial\mathcal{M}(u, v)$.** Left: the embedding in $\partial(\mathbb{R}_+^2 \times \mathbb{R}^{3d})$. Right: the corresponding embedding in $\mathbb{R} \times \mathbb{R}^3$ obtained by flattening the corner.

Proof. We frame the cube flow category in [LS, Proposition 4.12] inductively: We start with coherent framings of all moduli spaces of dimension less than k ; after changing the framings in the interior of the $(k - 1)$ -dimensional moduli spaces if necessary, we extend this to a framing of all k -dimensional moduli spaces.

Therefore, in order to prove this lemma, we merely need to check that the framings of the zero- and one-dimensional moduli spaces from Definition 3.4 can be extended to a framing of the two-dimensional moduli spaces.

Fix $v \leq_3 u$, and fix a neat embedding of the hexagon $\mathcal{M}(u, v)$ in $\mathbb{R}_+^2 \times \mathbb{R}^{3d}$. Let $[\bar{e}_1, \bar{e}_2, e_{11}, \dots, e_{1d}, e_{21}, \dots, e_{2d}, e_{31}, \dots, e_{3d}]$ be the standard basis for $\mathbb{R}_+^2 \times \mathbb{R}^{3d}$. The boundary K is a framed 6-gon embedded in $\partial(\mathbb{R}_+^2 \times \mathbb{R}^{3d})$. Let us flatten the corner in $\partial(\mathbb{R}_+^2 \times \mathbb{R}^{3d})$ so that $[\bar{e}_1 = -\bar{e}_2, e_{11}, \dots, e_{3d}]$ is the standard basis in the flattened $\mathbb{R} \times \mathbb{R}^{3d}$. After this flattening operation, we can treat K as a framed 1-manifold in $\mathbb{R} \times \mathbb{R}^{3d}$ —see Figure 3.3—which in turn represents some element $\eta_{u,v} \in \pi_4(S^3) = \mathbb{Z}/2$ by the Pontrjagin-Thom correspondence. We want to show that K is null-concordant, i.e., that $\eta_{u,v} = 0$.

As in Subsection 3.2, K can also be treated as a loop in $SO(3d + 1)$, and thus represents some element $h_{u,v} \in H_1(SO(3d + 1); \mathbb{Z}) = \mathbb{F}_2$. The element $h_{u,v}$ is non-zero if and only if $\eta_{u,v}$ is zero; therefore, we want to show $h_{u,v} = 1$.

Let $t_1, t_2, t_3, w_1, w_2, w_3$ be the six vertices between u and v in the cube, with $w_1 \leq_1 t_1, t_2$ and $w_2 \leq_1 t_1, t_3$ and $w_3 \leq_1 t_2, t_3$. For $i \in \{1, 2, 3\}$, let $s(\mathcal{C}_{u, t_i}) = a_i$, $s(\mathcal{C}_{w_i, v}) = c_i$, $f(\mathcal{C}_{u, w_i}) = f_i$ and $f(\mathcal{C}_{t_i, v}) = g_i$. Finally let $s(\mathcal{C}_{t_1, w_1}) = b_1$, $s(\mathcal{C}_{t_1, w_2}) = b_2$, $s(\mathcal{C}_{t_2, w_1}) = b_3$, $s(\mathcal{C}_{t_2, w_3}) = b_4$, $s(\mathcal{C}_{t_3, w_2}) = b_5$ and $s(\mathcal{C}_{t_3, w_3}) = b_6$. This information is encoded in the first part of Figure 3.4.

Consider the tri-colored planar graph G in the second part of Figure 3.4. The vertices represent frames in $\{0\} \times \mathbb{R}^{3d}$ as follows: if a vertex is labeled cba , then it represents the

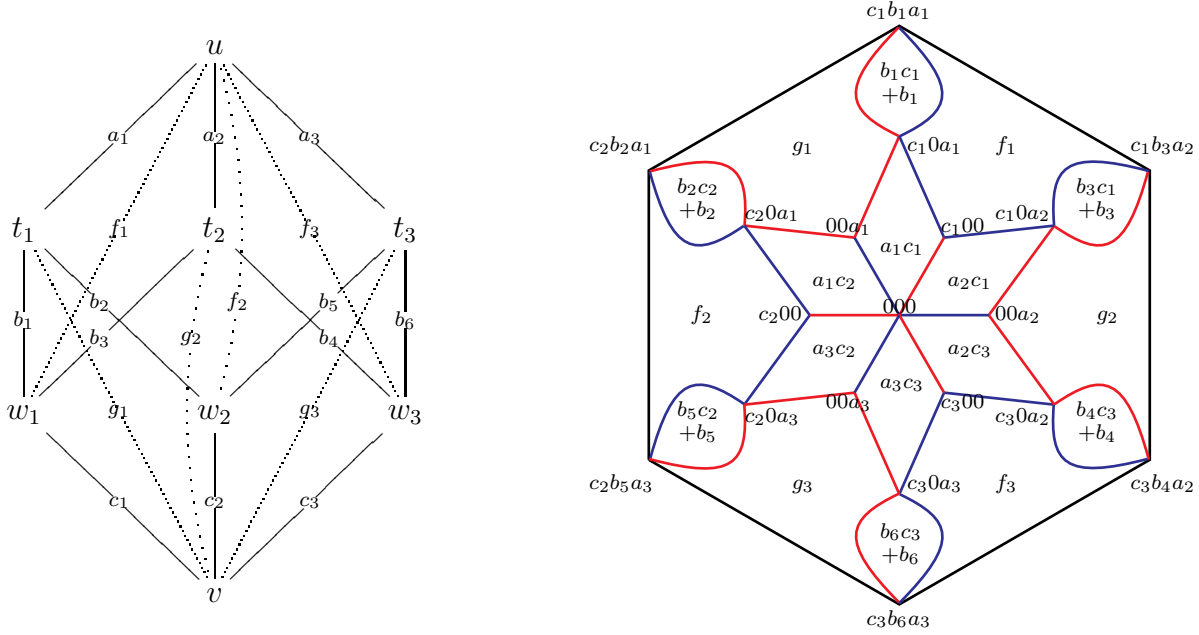


FIGURE 3.4. **Framing on the hexagon.** Left: notation for the framings at the vertices and edges of the hexagon. Right: the graph G .

frame

$$(0, (-1)^c e_{11}, e_{12}, \dots, e_{1d}, (-1)^b e_{21}, e_{22}, \dots, e_{2d}, (-1)^a e_{31}, e_{32}, \dots, e_{3d}).$$

Each edge represents a frame path joining the frames at its endpoints as follows.

- If the edge is colored black, i.e., if it is at the boundary of the hexagon, then it is one of the edges of the framed 6-cycle K .
- If the edge is colored red, then it represents the image under flattening of the following path in $\{0\} \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$: it is constant on the first \mathbb{R}^d and is a standard frame path on the remaining $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$.
- If the edge is colored blue, then it represents the image under flattening of the following path in $\mathbb{R}_+ \times \{0\} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$: it is constant on the last \mathbb{R}^d and is a standard frame path on the remaining $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$.

The element $h_{u,v} \in H_1(SO(3d+1))$ is represented by the black 6-cycle in G . In order to compute $h_{u,v}$, we will compute the homology classes of some other cycles in G .

Consider the black-blue 5-cycle joining $c_1b_1a_1$, c_10a_1 , c_100 , c_10a_2 and $c_1b_3a_2$. Modulo extending by the constant map on \mathbb{R}^d , the four blue edges represent standard frame paths in $\mathbb{R} \times \mathbb{R}^{2d}$ and the black edge is standard if and only if $f_1 = 0$. Therefore, this cycle represents

the element $f_1 \in H_1(SO(3d+1))$. We denote this by writing f_1 in the pentagonal region bounded by this 5-cycle in G . The homology classes represented by the other 5-cycles are shown in Figure 3.4.

Next consider the red-blue 4-cycle connecting $c_1 0a_1$, $00a_1$, 000 and $c_1 00$. If $a_1 = 0$, then the blue edges represent the constant paths, and the two red edges represent the same path; therefore the cycle is null-homologous. Similarly, if $c_1 = 0$, the cycle is null-homologous as well. Finally, if $a_1 = c_1 = 1$, it is easy to check from the definition of standard paths (Subsection 3.2) that the cycle represents the generator of $H_1(SO(3d+1))$. Therefore, the cycle represents the element $a_1 c_1$. The contributions from such 4-cycles is also shown in Figure 3.4.

Finally, consider the red-blue 2-cycle connecting $c_1 b_1 a_1$ and $c_1 0a_1$. If $b_1 = 0$, then both the red and the blue edges represent the constant paths, and hence the cycle is null-homologous. So let us concentrate on the case when $b_1 = 1$. Let

$$\begin{aligned}\varphi_1 &= (0, (-1)^{c_1} e_{11}, \dots, e_{1d}, -e_{21}, \dots, e_{2d}, (-1)^{a_1} e_{31}, \dots, e_{3d}) \text{ and} \\ \varphi_2 &= (0, (-1)^{c_1} e_{11}, \dots, e_{1d}, e_{21}, \dots, e_{2d}, (-1)^{a_1} e_{31}, \dots, e_{3d})\end{aligned}$$

be the two frames in $\{0\} \times \mathbb{R}^{3d}$. The blue edge represents the path from φ_1 to φ_2 where the $(d+2)^{\text{th}}$ vector rotates 180° in the $\langle \bar{e}_1, e_{21} \rangle$ -plane and equals $\bar{e}_2 = -\bar{e}_1$ halfway through. The red edge also represents a path where the $(d+2)^{\text{th}}$ vector rotates 180° in the $\langle \bar{e}_1, e_{21} \rangle$ -plane. However, halfway through, it equals \bar{e}_1 if $c_1 = 0$, and it equals $-\bar{e}_1$ if $c_1 = 1$. Hence, when $b_1 = 0$, the red-blue 2-cycle is null-homologous if and only if $c_1 = 1$. Therefore, the cycle represents the element $b_1(c_1 + 1) = b_1 c_1 + b_1$. These contributions are also shown in Figure 3.4.

We end the proof with a mild exercise in addition.

$$\begin{aligned}h_{u,v} &= (f_1 + f_2 + f_3 + g_1 + g_2 + g_3) + (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) \\ &\quad + (b_1 c_1 + b_3 c_1 + b_4 c_3 + b_6 c_3 + b_5 c_2 + b_2 c_2) + (a_1 c_1 + a_2 c_1 + a_2 c_3 + a_3 c_3 + a_3 c_2 + a_1 c_2) \\ &= (c_1 + c_2 + c_3) + (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) \quad (\text{by Lemma 2.1}) \\ &\quad + (c_1(a_1 + a_2 + b_1 + b_3) + c_2(a_1 + a_3 + b_2 + b_5) + c_3(a_2 + a_3 + b_4 + b_6)) \\ &= (c_1 + c_2 + c_3) + (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) \\ &\quad + (c_1 + c_2 + c_3) \quad (\text{since } s \text{ is a sign assignment}) \\ &= (b_1 + b_2 + c_1 + c_2) + (b_3 + b_4 + c_1 + c_3) + (b_5 + b_6 + c_2 + c_3) \\ &= 1 + 1 + 1 \quad (s \text{ is still a sign assignment}) \\ &= 1. \quad \square\end{aligned}$$

3.4. Sq² for the Khovanov homotopy type. Fix a link diagram L and an integer ℓ , and let $\mathcal{X}_{Kh}^\ell(L)$ denote the Khovanov homotopy type constructed in [LS]. We want to study the

Steenrod square

$$\mathrm{Sq}^2: \widetilde{H}^\kappa(\mathcal{X}_{Kh}^\ell(L)) \rightarrow \widetilde{H}^{\kappa+2}(\mathcal{X}_{Kh}^\ell(L)).$$

The spectrum $\mathcal{X}_{Kh}^\ell(L)$ is a formal de-suspension $\Sigma^{-N}Y_\ell$ of a CW complex Y_ℓ for some sufficiently large N . Therefore, we want to understand the Steenrod square

$$\mathrm{Sq}^2: H^{N+\kappa}(Y_\ell; \mathbb{F}_2) \rightarrow H^{N+\kappa+2}(Y_\ell; \mathbb{F}_2).$$

Before we get started, we give names to a few maps which will make regular appearances. Fix $m_1, m_2 \geq 2$ and let $m = m_1 + m_2$. First, recall that we have maps $\mathfrak{r}, \mathfrak{s}: \mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}^m$ given by Formulas (3.1) and (3.2), respectively. Next, let

$$\Pi: \mathbb{D}^m \rightarrow K_m^{(m+1)}$$

be the composition of the projection map $\mathbb{D}^m \rightarrow \mathbb{D}^m / \partial \mathbb{D}^m = S^m$ and the inclusion $S^m \rightarrow K_m^{(m+1)}$. Let

$$\Xi: [0, 1] \times \mathbb{D}^m \rightarrow K_m^{(m+1)}$$

be the map induced by the identification of $[0, 1] \times \mathbb{D}^m$ with the $(m+1)$ -cell e^{m+1} of $K_m^{(m+1)}$; the map Ξ collapses $[0, 1] \times \partial \mathbb{D}^m$ to the basepoint and maps each of $\{0\} \times \mathbb{D}^m$ and $1 \times \mathbb{D}^m$ to the m -skeleton $S^m \subset K_m^{(m+1)}$ by Π and $\Pi \circ \mathfrak{r}$, respectively. The map Ξ factors through a map

$$\bar{\Xi}: [0, 1] \times S^m \rightarrow K_m^{(m+1)}.$$

From this data, we construct a CW complex X as follows. We choose real numbers ϵ and R with $0 < \epsilon \ll R$. Then:

Step 1: Start with a unique 0-cell e^0 .

Step 2: For each Khovanov generator $\mathfrak{x}_i \in KG^{\kappa, \ell}$, X has a corresponding cell

$$f_i^m = \{0\} \times \{0\} \times [-\epsilon, \epsilon]^{m_1} \times [-\epsilon, \epsilon]^{m_2}.$$

The boundary of f_i^m is glued to e^0 .

Step 3: For each Khovanov generator $\mathfrak{x}_j \in KG^{\kappa+1, \ell}$, X has a corresponding cell

$$f_j^{m+1} = [0, R] \times \{0\} \times [-R, R]^{m_1} \times [-\epsilon, \epsilon]^{m_2}.$$

The boundary of f_j^{m+1} is attached to $X^{(m)}$ as follows. If \mathfrak{x}_j occurs in $\delta_{\mathbb{Z}} \mathfrak{x}_i$ with sign $\epsilon_{i,j} \in \{\pm 1\}$ then we embed f_i^m in ∂f_j^{m+1} by a map of the form

$$(3.3) \quad \begin{aligned} & (0, 0, x_1, x_2, \dots, x_{m_1}, y_1, \dots, y_{m_2}) \\ & \mapsto (0, 0, \epsilon_{i,j} x_1 + a_1, x_2 + a_2, \dots, x_{m_1} + a_{m_1}, y_1, \dots, y_{m_2}) \end{aligned}$$

for some vector (a_1, \dots, a_{m_1}) . Call the image of this embedding $\mathcal{C}_i(j)$. Then $\mathcal{C}_i(j)$ is mapped to f_i^m by the specified identification, and $(\partial f_j^{m+1}) \setminus \bigcup_i \mathcal{C}_i(j)$ is mapped to the basepoint e^0 .

We choose the vectors (a_1, \dots, a_{m_1}) so that the different $\mathcal{C}_i(j)$'s are disjoint. Write $p_{i,j} = (0, a_1, \dots, a_{m_1}, 0, \dots, 0) \in f_j^{m+1}$.

Step 4: For each Khovanov generator $\mathbf{x}_k \in KG^{\kappa+2,\ell}$, X has a corresponding cell

$$f_k^{m+2} = [0, R] \times [0, R] \times [-R, R]^{m_1} \times [-R, R]^{m_2}.$$

The boundary of f_k^{m+2} is attached to $X^{(m+1)}$ as follows. First, we choose some auxiliary data:

- If \mathbf{x}_k occurs in $\delta_{\mathbb{Z}}\mathbf{x}_j$ with sign $\epsilon_{j,k} \in \{\pm 1\}$ then we embed f_j^{m+1} in ∂f_k^{m+2} by a map of the form
- $$(3.4) \quad \begin{aligned} & (x_0, 0, x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}) \\ & \mapsto (x_0, 0, x_1, x_2, \dots, x_{m_1}, 0, \epsilon_{j,k}y_1 + b_1, \dots, y_{m_2} + b_{m_2}) \end{aligned}$$

for some vector (b_1, \dots, b_{m_2}) . Call the image of this embedding $\mathcal{C}_j(k)$. Once again, we choose the vectors (b_1, \dots, b_{m_2}) so that the different $\mathcal{C}_j(k)$'s are disjoint.

- Let $\mathbf{x}_i \in KG^{\kappa,\ell}$. If the set of generators $\mathcal{G}_{\mathbf{x}_k, \mathbf{x}_i}$ between \mathbf{x}_k and \mathbf{x}_i is nonempty then $\mathcal{G}_{\mathbf{x}_k, \mathbf{x}_i}$ consists of 2 or 4 points, and these points are identified in pairs (via the ladybug matching of Subsection 2.4). Write $\mathcal{G}_{\mathbf{x}_k, \mathbf{x}_i} = \{\mathbf{x}_{j_\beta}\}$. For each j_β , the cell $\mathcal{C}_i(j_\beta)$ can be viewed as lying in the boundary of $\mathcal{C}_{j_\beta}(k)$. Consider the point p_{i,j_β} in the interior of $\mathcal{C}_i(j_\beta) \subset \partial \mathcal{C}_{j_\beta}(k)$. Each of the points p_{i,j_β} inherits a framing, i.e., a trivialization of the normal bundle to p_{i,j_β} in $\partial \mathcal{C}_{j_\beta}(k)$, from the map $f_i^m \rightarrow \partial f_k^{m+2}$,

$$\begin{aligned} & (0, 0, x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}) \\ & \mapsto (0, 0, \epsilon_{i,j_\beta}x_1 + a_1, x_2 + a_2, \dots, x_{m_1} + a_{m_1}, \epsilon_{j_\beta,k}y_1 + b_1, \dots, y_{m_2} + b_{m_2}). \end{aligned}$$

Notice that the framing of p_{i,j_β} is one of the standard frames for $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ (Definition 3.2).

The pair of generators $(\mathbf{x}_i, \mathbf{x}_k)$ specifies a 2-dimensional face of the hypercube $\mathcal{C}(n)$. Let $\mathcal{C}_{k,i}$ denote this face. The standard frame assignment f of Subsection 2.1 assigns an element $f(\mathcal{C}_{k,i}) \in \mathbb{F}_2$ to the face $\mathcal{C}_{k,i}$.

The matching of elements of $\mathcal{G}_{\mathbf{x}_k, \mathbf{x}_i}$ matches the points p_{i,j_a} in pairs. Moreover, it follows from the definition of the sign assignment that matched pairs of points have opposite framings. For each matched pair of points choose a properly embedded arc

$$\zeta \subset \{0\} \times [0, R] \times [-R, R]^{m_1} \times [-R, R]^{m_2} \subset \partial f_k^{m+2}$$

connecting the pair of points. The endpoints of ζ are framed. Extend this to a framing of the normal bundle to ζ in ∂f_k^{m+2} . If $f(\mathcal{C}_{k,i}) = 0$, then choose this framing to be isotopic relative boundary to a standard frame path for $\{0\} \times \mathbb{R} \times \mathbb{R}^m$; if $f(\mathcal{C}_{k,i}) = 1$, then choose this framing to be isotopic relative boundary to a non-standard frame path for $\{0\} \times \mathbb{R} \times \mathbb{R}^m$.

We call these arcs ζ *Pontrjagin-Thom arcs*, and denote the set of them by $\{\zeta_{i_1,1}, \dots, \zeta_{i_A,n_A}\}$ where the arc $\zeta_{i_\alpha,i}$ comes from the generator $\mathbf{x}_{i_\alpha} \in KG^{\kappa,\ell}$.

The choice of these auxiliary data is illustrated in Figure 3.5. Now, the attaching map on ∂f_k^{m+2} is given as follows:

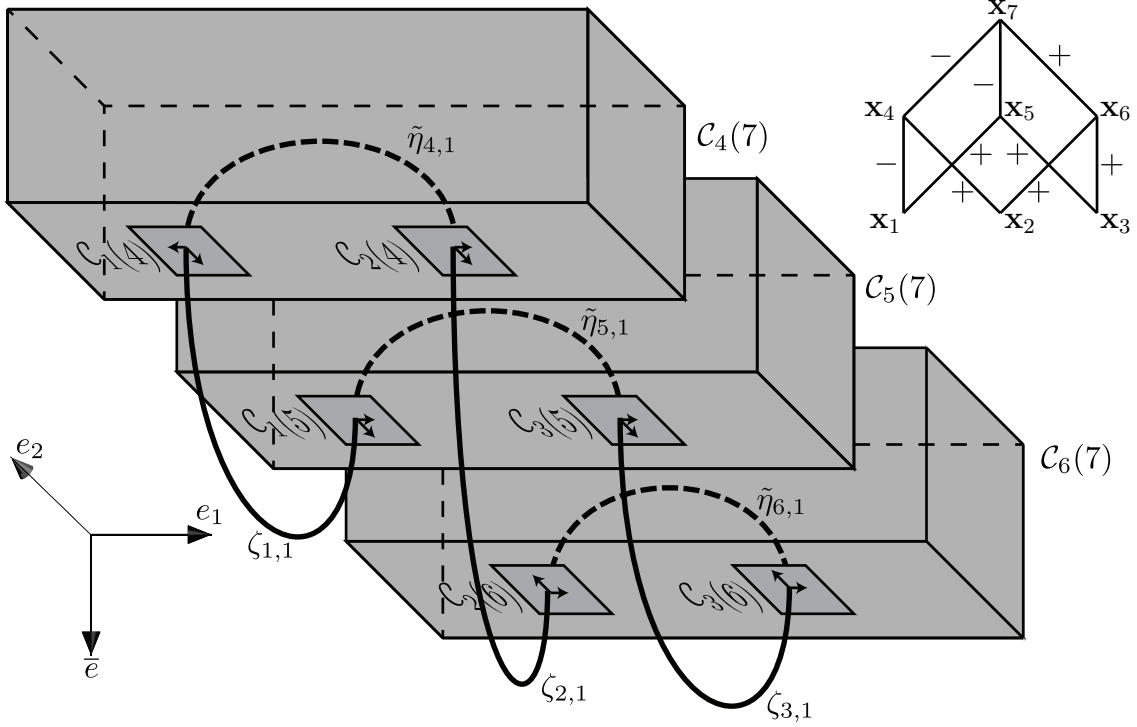


FIGURE 3.5. **The attaching map corresponding to a generator x_7 .** The boundary matching arcs are also shown; $\tilde{\eta}_{4,1}$ is boundary-coherent while $\tilde{\eta}_{5,1}$ and $\tilde{\eta}_{6,1}$ are boundary-incoherent. Here, $m_1 = m_2 = 1$ and we have identified $([0, R] \times \{0\} \times [-R, R] \times [-R, R]) \cup (\{0\} \times [0, R] \times [-R, R] \times [-R, R]) \cong [-R, R] \times [-R, R] \times [-R, R]$ via a reflection on the first summand and the identity map on the second summand.

- The interior of $\mathcal{C}_j(k)$ is mapped to f_j^{m+1} by (the inverse of) the identification in Formula (3.4).
- A tubular neighborhood of each Pontrjagin-Thom arc $\text{nb}d(\zeta_{i_\alpha, i})$ is mapped to $f_{i_\alpha}^m$ as follows. The framing identifies

$$\text{nb}d(\zeta_{i_\alpha, i}) \cong \zeta_{i_\alpha, i} \times [-\epsilon, \epsilon]^{m_1+m_2} \cong \zeta_{i_\alpha, i} \times f_{i_\alpha}^m.$$

With respect to this identification, the map is the obvious projection to $f_{i_\alpha}^m$.

- The rest of ∂f_k^{m+2} is mapped to the basepoint e^0 .

Proposition 3.6. *Let X denote the space constructed above and $Y_\ell = Y_\ell(L)$ the CW complex from [LS] associated to L in quantum grading ℓ . Then*

$$\Sigma^{N+\kappa-m} X = Y_\ell^{(N+\kappa+2)} / Y_\ell^{(N+\kappa-1)}.$$

Proof. The construction of X above differs from the construction of Y_ℓ in [LS] as follows:

- (1) We have collapsed all cells of dimension less than m to the basepoint, and ignored all cells of dimension bigger than $m + 2$.
- (2) In the construction above, we have suppressed the 0-dimensional framed moduli spaces, instead speaking directly about the embeddings of cells that they induce. The 0-dimensional moduli spaces correspond to the points (a_1, \dots, a_{m_1}) and (b_1, \dots, b_{m_2}) in Formulas (3.3) and (3.4), respectively. Their framings are induced by the maps in Formulas (3.3) and (3.4).
- (3) In the construction of [LS, Definition 3.23], each of the cells above were multiplied by $[0, R]^{p_1} \times [-R, R]^{p_2} \times [-\epsilon, \epsilon]^{p_3}$ for some $p_1, p_2, p_3 \in \mathbb{N}$ with $p_1 + p_2 + p_3 = N + \kappa - m$; and the various multiplicands were ordered differently. This has the effect of suspending the space X by $(N + \kappa - m)$ -many times.
- (4) The framings in [LS] were given by an obstruction-theory argument [LS, Proposition 4.12], while the framings here are given explicitly by the standard sign assignment and standard frame assignment. This is justified by Lemma 3.5.

Thus, up to stabilizing, the two constructions give the same space. \square

Therefore, it is enough to study the Steenrod square $\text{Sq}^2: H^m(X; \mathbb{F}_2) \rightarrow H^{m+2}(X; \mathbb{F}_2)$. Fix a cohomology class $[c] \in H^m(X; \mathbb{F}_2)$. Let

$$c = \sum_{f_i^m} c_i (f_i^m)^*$$

be a cocycle representing $[c]$. Here, the c_i are elements of $\{0, 1\}$.

We want to understand the map $\mathbf{c}: X \rightarrow K_m^{(m+2)}$ corresponding to c . We start with $\mathbf{c}^{(m)}: X^{(m)} \rightarrow K_m^{(m)}$: on f_i^m , this map is defined as follows:

- the projection Π composed with the identification $[-\epsilon, \epsilon]^m = \mathbb{D}^m$ if $c_j = 1$.
- the constant map to the basepoint of $K_m^{(m)}$ if $c_j = 0$.

To extend \mathbf{c} to $X^{(m+1)}$ we need to make one more auxiliary choice:

Definition 3.7. A *topological boundary matching* for c consists of the following data for each $(m + 1)$ -cell f_j^{m+1} : a collection of disjoint, embedded, framed arcs $\eta_{j,j}$ in f_j^{m+1} connecting the points

$$\coprod_{i|c_i=1} p_{i,j} \subset \partial f_j^{m+1}$$

in pairs, together with framings of the normal bundles to the $\eta_{j,j}$.

The normal bundle in f_j^{m+1} to each of the points $p_{i,j}$ inherits a framing from Formula (3.3). Call an arc $\eta_{j,j}$ *boundary-coherent* if the points $p_{i_1,j}$ and $p_{i_2,j}$ in $\partial \eta_{j,j}$ have opposite framings, i.e., if $\epsilon_{i_1,j} = -\epsilon_{i_2,j}$, and *boundary-incoherent* otherwise. We require the following conditions on the framings for the $\eta_{j,j}$:

- Trivialize Tf_j^{m+1} using the following inclusion

$$f_j^{m+1} = [0, R] \times \{0\} \times [-R, R]^{m_1} \times [-\epsilon, \epsilon]^{m_2} \hookrightarrow \mathbb{R}^{m+1}$$

$$(t, 0, x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}) \mapsto (-t, x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}).$$

We require the framing of $\eta_{j,j}$ to be isotopic relative boundary to one of the standard frame paths for $\mathbb{R} \times \mathbb{R}^m$.

- If $\eta_{j,j}$ is boundary-coherent then the framing of $\eta_{j,j}$ is compatible with the framing of its boundary.
- If $\eta_{j,j}$ is boundary-incoherent then the framing of one end of $\eta_{j,j}$ agrees with the framing of the corresponding $p_{i_1,j}$ while the framing of the other end of $\eta_{j,j}$ differs from the framing of $p_{i_2,j}$ by the reflection $\mathbf{r}: \mathbb{R}^m \rightarrow \mathbb{R}^m$.

Each boundary-incoherent arc in a topological boundary matching inherits an orientation: it is oriented from the endpoint $p_{i,j}$ at which the framings agree to the endpoint at which the framings disagree.

Lemma 3.8. *A topological boundary matching for c exists.*

Proof. Since c is a cocycle, for each $(m+1)$ -cell f_j^{m+1} we have

$$\sum_i \epsilon_{i,j} c_i = c(\sum_i \epsilon_{i,j} f_i^m) = c(\partial f_j^{m+1}) \equiv 0 \pmod{2}.$$

Together with our condition on m , this ensures that a topological boundary matching for c exists. \square

The map $\mathbf{c}^{(m+1)}: X^{(m+1)} \rightarrow K_m^{(m+1)}$ is defined using the topological boundary matching as follows. On f_j^{m+1} :

- The map $\mathbf{c}^{(m+1)}$ sends the complement of a neighborhood of the $\eta_{j,j}$ to the basepoint.
- If $\eta_{j,j}$ is boundary-coherent then the framing of $\eta_{j,j}$ identifies a neighborhood of the arc $\eta_{j,j}$ with $\eta_{j,j} \times \mathbb{D}^m$. With respect to this identification, the map $\mathbf{c}^{(m+1)}$ is projection $\eta_{j,j} \times \mathbb{D}^m \rightarrow \mathbb{D}^m \xrightarrow{\Pi} K_m^{(m+1)}$.

Note that $\mathbf{c}^{(m)}$ induces a map $(\partial\eta_{j,j}) \times \mathbb{D}^m$, and that the compatibility condition of the framing of $\eta_{j,j}$ with the framing of $\partial\eta_{j,j}$ implies that the map $\mathbf{c}^{(m+1)}$ extends the map $\mathbf{c}^{(m)}$.

- If $\eta_{j,j}$ is boundary-incoherent then the orientation and framing identify a neighborhood of $\eta_{j,j}$ with $[0, 1] \times \mathbb{D}^m$. With respect to this identification, $\mathbf{c}^{(m+1)}$ is given by the map Ξ .

Again, $\mathbf{c}^{(m)}$ induces a map $\{0, 1\} \times \mathbb{D}^m$, and the compatibility condition of the framing of $\eta_{j,j}$ with the framing of $\partial\eta_{j,j}$ implies that the map $\mathbf{c}^{(m+1)}$ extends the map $\mathbf{c}^{(m)}$.

Now, fix an $(m+2)$ -cell f^{m+2} . We want to compute the element

$$\mathfrak{c}|_{\partial f^{m+2}} \in \pi_{m+1}(K_m^{(m+1)}) = \mathbb{Z}/2.$$

As described above,

$$\begin{aligned} \partial f^{m+2} = & (\{0\} \times [0, R] \times [-R, R]^m) \cup (\{R\} \times [0, R] \times [-R, R]^m) \\ & \cup ([0, R] \times \{0\} \times [-R, R]^m) \cup ([0, R] \times \{R\} \times [-R, R]^m) \end{aligned}$$

has corners. The map $\mathfrak{c}|_{\partial f^{m+2}}$ will send

$$(\{R\} \times [0, R] \times [-R, R]^m) \cup ([0, R] \times \{R\} \times [-R, R]^m)$$

to the basepoint. We straighten the corner between the other two parts of ∂f^{m+2} via the map

$$\begin{aligned} (\{0\} \times [0, R] \times [-R, R]^m) \cup ([0, R] \times \{0\} \times [-R, R]^m) & \rightarrow [-R, R] \times [-R, R]^m \\ (0, t, x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}) & \mapsto (t, x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}) \\ (t, 0, x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}) & \mapsto (-t, x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}). \end{aligned}$$

We will suppress this straightening from the notation in the rest of the section.

Let $\zeta_1, \dots, \zeta_k \subset S^{m+1} = \partial f^{m+2}$ be the Pontrjagin-Thom arcs corresponding to f . Let $\{\tilde{\eta}_{j,j}\}$ be the preimages in $S^{m+1} = \partial f^{m+2}$ of the topological boundary matching. The union

$$\bigcup_{j,j} \tilde{\eta}_{j,j} \cup \bigcup_i \zeta_i$$

is a one-manifold in S^{m+1} . Each of the arcs $\zeta_i \subset \partial f^{m+2}$ comes with a framing. Each of the arcs $\tilde{\eta}_{j,j} \subset \partial f^{m+2}$ also inherits a framing: the pushforward of the framing of $\eta_{j,j}$ under the map of Formula (3.4). The map $\mathfrak{c}|_{\partial f^{m+2}} : S^{m+1} \rightarrow K_m^{(m+1)}$ is induced from these framed arcs as follows:

- A tubular neighborhood of each Pontrjagin-Thom arc ζ_i is mapped via

$$\text{nb}(\zeta_i) \cong \zeta_i \times \mathbb{D}^m \rightarrow \mathbb{D}^m \xrightarrow{\Pi} S^m,$$

where the first isomorphism is induced by the framing.

- A tubular neighborhood of each boundary-coherent $\tilde{\eta}_{j,j}$ is mapped via

$$\text{nb}(\tilde{\eta}_{j,j}) \cong \tilde{\eta}_{j,j} \times \mathbb{D}^m \rightarrow \mathbb{D}^m \xrightarrow{\Pi} S^m,$$

where the first isomorphism is induced by the framing.

- A tubular neighborhood of each boundary-incoherent $\tilde{\eta}_{j,j}$ is mapped via

$$\text{nb}(\tilde{\eta}_{j,j}) \cong [0, 1] \times \mathbb{D}^m \xrightarrow{\Xi} K_m^{(m+1)}.$$

where the first isomorphism is induced by the orientation and framing.

- The map \mathfrak{c} takes the rest of S^{m+1} to the basepoint of $K_m^{(m+1)}$.

Let K be a component of $\bigcup_i \zeta_i \cup \bigcup_{j,j'} \tilde{\eta}_{j,j'}$. Relabeling, let p_1, \dots, p_{2k} be the points $p_{i,j}$ on K , $\tilde{\eta}_1, \dots, \tilde{\eta}_k$ the sub-arcs of K coming from the topological boundary matching and ζ_1, \dots, ζ_k the sub-arcs of K coming from the Pontrjagin-Thom data. Order these so that $\partial \zeta_i = \{p_{2i-1}, p_{2i}\}$ and $\partial \tilde{\eta}_i = \{p_{2i}, p_{2i+1}\}$.

We define an isomorphism $\Phi: \text{nbid}(K) \rightarrow K \times \mathbb{D}^m$ as follows. First, the framing of ζ_1 induces an identification of the normal bundle $N_{p_1} K$ with \mathbb{D}^m . Second, the framing of each arc $\gamma \in \{\zeta_i, \tilde{\eta}_i\}$ induces a trivialization of the normal bundle $N\gamma$. Suppose that the framing of K has already been defined at the endpoint p_i of γ . Then the trivialization of $N\gamma$ allows us to transport the framing of p_i along γ . This transported framing is the framing of K along γ .

Note that the framing of K along γ may not agree with the original framing of γ ; but the two either agree or differ by the map

$$K \times \mathbb{D}^m \rightarrow K \times \mathbb{D}^m \quad (x, v) \mapsto (x, \mathfrak{r}(v)).$$

Call an arc γ in K \mathfrak{r} -colored if the original framing of γ disagrees with the framing of K , and Id-colored if the original framing of γ agrees with the framing of K .

Write $\Psi = \mathfrak{c} \circ \Phi^{-1}: K \times \mathbb{D}^m \rightarrow K_m^{(m+1)}$. Explicitly, the map Ψ is given as follows:

- If γ_i is one of the Pontrjagin-Thom arcs or is a boundary-coherent topological boundary matching arc then a neighborhood of γ_i in $K \times \mathbb{D}^m$ is mapped to $S^m \subset K_m^{(m+1)}$ by the map

$$\begin{aligned} \gamma_i \times \mathbb{D}^m \ni (x, v) &\mapsto \Pi(v) \in K_m^{(m+1)} && \text{if } \gamma_i \text{ is Id-colored} \\ \gamma_i \times \mathbb{D}^m \ni (x, v) &\mapsto \Pi(\mathfrak{r}(v)) \in K_m^{(m+1)} && \text{if } \gamma_i \text{ is } \mathfrak{r}\text{-colored.} \end{aligned}$$

- If $\tilde{\eta}_i$ is boundary-incoherent then the framing of K and the orientation of $\tilde{\eta}_i$ induce an identification $\text{nbid}(\tilde{\eta}_i) \cong [0, 1] \times \mathbb{D}^m$. With respect to this identification, $\text{nbid}(\tilde{\eta}_i)$ is mapped to $K_m^{(m+1)}$ by the map

$$\begin{aligned} (t, v) &\mapsto \Xi(t, v) && \text{if } \tilde{\eta}_i \text{ is Id-colored} \\ (t, v) &\mapsto \Xi(t, \mathfrak{r}(v)) && \text{if } \tilde{\eta}_i \text{ is } \mathfrak{r}\text{-colored.} \end{aligned}$$

Let Ψ' be the projection $K \times \mathbb{D}^m \rightarrow \mathbb{D}^m \xrightarrow{\Pi} S^m$. These maps are summarized in the following diagram:

$$(3.5) \quad \begin{array}{ccccc} & & \text{c} & & \\ & \nearrow & & \searrow & \\ \text{nbid}(K) & \xrightarrow[\cong]{\Phi} & K \times \mathbb{D}^m & \xrightarrow{\Psi} & K_m^{(m+1)} \\ & & \searrow \Psi' & & \uparrow \iota \\ & & & & S^m \end{array}$$

It is immediate from the definitions that the top triangle commutes. Our next goal is to show that the other triangle commutes up to homotopy:

Proposition 3.9. *The map Ψ is homotopic relative $(\partial K \times \mathbb{D}^m) \cup (\{p_1\} \times \mathbb{D}^m)$ to $\iota \circ \Psi'$, i.e., the bottom triangle of Diagram (3.5) commutes up to homotopy relative $(\partial K \times \mathbb{D}^m) \cup (\{p_1\} \times \mathbb{D}^m)$.*

The proof of Proposition 3.9 uses a model computation. Consider the map $\Xi: [0, 1] \times S^m \rightarrow K_m^{(m+1)}$. Concatenation in $[0, 1]$ endows $\text{Hom}([0, 1] \times S^m, K_m^{(m+1)})$ with a multiplication, which we denote by $*$. Let $\mathbf{t}: [0, 1] \rightarrow [0, 1]$ be the reflection $\bar{f}(t, x) = f(1 - t, x)$. Using \mathbf{t} , we obtain a map $\Xi \circ (\mathbf{t} \times \text{Id}): [0, 1] \times S^m \rightarrow K_m^{(m+1)}$. Finally, using the map \mathbf{r} we obtain a map $\Xi \circ (\text{Id} \times \mathbf{r}): [0, 1] \times S^m \rightarrow K_m^{(m+1)}$.

Lemma 3.10. *Assume that $m \geq 3$. Then both $\Xi * [\Xi \circ (\mathbf{t} \times \text{Id})]$ and $\Xi * [\Xi \circ (\text{Id} \times \mathbf{r})]$ are homotopic (relative boundary) to the map $[0, 1] \times S^m \rightarrow S^m \subset K_m^{(m+1)}$ given by $(t, x) \mapsto x$ (i.e., the constant path in $O(m+1)$ with value Id).*

Proof. The statement about $\Xi * [\Xi \circ (\mathbf{t} \times \text{Id})]$ is obvious. For the statement about $\Xi * [\Xi \circ (\text{Id} \times \mathbf{r})]$, let $H_m = \Xi * [\Xi \circ (\text{Id} \times \mathbf{r})]: [0, 1] \times S^m \rightarrow K_m^{(m+1)}$. We can view H_m as an element of $\pi_1(\Omega^m K_m^{(m+1)}) \cong \pi_{m+1}(K_m^{(m+1)})$. Moreover, the map H_m is the $(m-1)$ -fold suspension of the map $H_1: [0, 1] \times S^1 \rightarrow K_1^{(2)} = \mathbb{R}P^2$. But the suspension map

$$\Sigma^i: \pi_2(\mathbb{R}P^2) \rightarrow \pi_{i+2}(\Sigma^i \mathbb{R}P^2)$$

is nullhomotopic for $i \geq 2$; see, for instance, [Wu03, Proposition 6.5 and discussion before Proposition 6.11]. So, it follows from our assumption on m that $H_1 \in \pi_1(\Omega^m K_m^{(m+1)})$ is homotopically trivial. Keeping in mind that our loops are based at the identity map $S^m \rightarrow S^m \subset K_m^{(m+1)}$, this proves the result. \square

Lemma 3.11. *An even number of the arcs in K are boundary-incoherent.*

Proof. The proof is essentially the same as the proof of the second half of Lemma 2.7, and is left to the reader. \square

Proof of Proposition 3.9. As in the proof of Lemma 3.10, we can view Ψ as an element of $\pi_1(\Omega^m K_m^{(m+1)}, \iota)$, i.e., a loop of maps $S^m \rightarrow K_m^{(m+1)}$ based at the map $\iota: S^m \rightarrow K_m^{(m+1)}$. From its definition, Ψ decomposes as a product of paths,

$$\Psi = \Psi_{\gamma_1} * \cdots * \Psi_{\gamma_{2k}},$$

one for each arc γ_i in K . Here, Ψ_{γ_i} is an element of the fundamental groupoid of $\Omega^m K_m^{(m+1)}$, with endpoints in $\{\iota, \iota \circ \mathbf{r}\}$. The path Ψ_{γ_i} is:

- The constant path based at either ι or $\iota \circ \mathbf{r}$ if γ_i is one of the Pontrjagin-Thom arcs or is a boundary-coherent topological boundary matching arc.

- One of the paths Ξ , $\Xi \circ (\text{Id} \times \mathfrak{r})$, $\Xi \circ (\mathfrak{t} \times \text{Id})$ or $\Xi \circ (\mathfrak{t} \times \text{Id}) \circ (\text{Id} \times \mathfrak{r})$ if γ_i is a boundary-incoherent topological boundary matching arc.

Contracting the constant paths, Ψ can be expressed as

$$\Psi = \Psi_{\eta_{i_1}} * \Psi_{\eta_{i_2}} * \cdots * \Psi_{\eta_{i_A}}$$

where the η_{i_α} are boundary-incoherent. By Lemma 3.11, A is even. Moreover, $\Psi_{\eta_{i_\alpha}}$ is either Ξ or $\Xi \circ (\mathfrak{t} \times \text{Id}) \circ (\text{Id} \times \mathfrak{r})$ if α is odd, and is either $\Xi \circ (\mathfrak{t} \times \text{Id})$ or $\Xi \circ (\text{Id} \times \mathfrak{r})$ if α is even. So, by Lemma 3.10, the concatenation $\Psi_{\eta_{i_{2\alpha-1}}} * \Psi_{\eta_{i_{2\alpha}}}$ is homotopic to the constant path ι . The result follows. \square

The pair (K, Φ) specifies a framed cobordism class $[K, \Phi] \in \Omega_1^{\text{fr}} = \mathbb{Z}/2$.

Proposition 3.12. *The element $[K, \Phi] \in \mathbb{Z}/2$ is given by the sum of:*

- (1) 1.
- (2) The number of Pontrjagin-Thom arcs in K with the non-standard framing.
- (3) The number of arrows on K which point in a given direction.

Proof. First, exchanging the standard and non-standard framings on an arc changes the overall framing of K by 1. So, it suffices to prove the proposition in the case that all of the Pontrjagin-Thom arcs in K have the standard framing.

Second, the framing on each boundary-matching arc is standard if the corresponding $(m+1)$ -cell occurs positively in ∂f^{m+2} , and differs from the standard framing by the map \mathfrak{s} of Subsection 3.2 if the $(m+1)$ -cell occurs negatively in ∂f^{m+2} . In the notation of Subsection 3.2, the framings of the boundary matching arcs are among $\{\overline{+-}, \overline{+-}, \overline{-+}, \overline{-+}\}_{++, --, ++, --}$. So, by Lemma 3.3, \mathfrak{s} takes the standard frame path on a boundary-matching arc to a standard frame path. In sum, each of the arcs $\tilde{\eta}_i$ is framed by a standard frame path.

So, by Lemma 3.3, the framing of K at each arc γ_i is standard if γ_i is Id-colored, and non-standard if γ_i is \mathfrak{r} -colored. Thus, it suffices to show that the number of \mathfrak{r} -colored arcs agrees modulo 2 with the number of arrows on K which point in a given direction.

Let $\tilde{\eta}_{i_1}, \dots, \tilde{\eta}_{i_A}$ be the boundary-incoherent boundary matching arcs in K . Then:

- There are an odd number of arcs strictly between $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$. Moreover:
 - These arcs are all \mathfrak{r} -colored if α is odd.
 - These arcs are all Id-colored if α is even.
- If $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ are oriented in the same direction then exactly one of $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ is \mathfrak{r} -colored.
- If $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ are oriented in opposite directions then either both of $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ are \mathfrak{r} -colored or both of $\tilde{\eta}_{i_\alpha}$ and $\tilde{\eta}_{i_{\alpha+1}}$ are Id-colored.

It follows that there are an even (respectively odd) number of \mathfrak{r} -colored arcs in the interval $[\tilde{\eta}_{i_{2\alpha-1}}, \tilde{\eta}_{i_{2\alpha}}]$ if $\tilde{\eta}_{i_{2\alpha-1}}$ and $\tilde{\eta}_{i_{2\alpha}}$ are oriented in the same direction (respectively opposite directions); and all of the arcs in the interval $(\tilde{\eta}_{i_{2\alpha}}, \tilde{\eta}_{i_{2\alpha+1}})$ are Id-colored. So, the number of \mathfrak{r} -colored arcs agrees with the number of arcs which point in a given direction.

Finally, the contribution 1 comes from the fact that the constant loop in $SO(m)$ corresponds to the nontrivial element of Ω_1^{fr} . \square

Proof of Theorem 2. Let Φ_K and Ψ_K be the maps associated above to each component K of $\bigcup_i \zeta_i \cup \bigcup_{j,j'} \tilde{\eta}_{j,j'}$. Then $\Psi_K \circ \Phi_K$ induces an element $[K, \Psi_K \circ \Phi_K]$ of $\Omega_1^{\text{fr}} = \mathbb{Z}/2$. Let \mathbf{x} be the generator of the Khovanov complex corresponding to the cell f^{m+2} and \mathbf{c} the element of the Khovanov chain group corresponding to the cocycle c . Then it suffices to show that the sum

$$\sum_K [K, \Psi_K \circ \Phi_K]$$

agrees with the expression

$$(3.6) \quad \left(\#|\mathfrak{G}_{\mathbf{c}}(\mathbf{x})| + f(\mathfrak{G}_{\mathbf{c}}(\mathbf{x})) + g(\mathfrak{G}_{\mathbf{c}}(\mathbf{x})) \right) \mathbf{x},$$

from Formula (2.3).

By Proposition 3.9, $[K, \Psi_K \circ \Phi_K] = [K, \Phi]$. The element $[K, \Phi] \in \mathbb{Z}/2$ is computed in Proposition 3.12, and it remains to match the terms in that proposition with the terms in Formula (3.6).

By construction, the graph $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ is exactly $\bigcup_i \zeta_i \cup \bigcup_{j,j'} \tilde{\eta}_{j,j'}$, and the orientations of the oriented edges of $\mathfrak{G}_{\mathbf{c}}(\mathbf{x})$ match up with the orientations of the boundary-incoherent $\tilde{\eta}_i$. So, the first term in Formula (3.6) corresponds to part (1) of Proposition 3.12, and the third term in Formula (3.6) corresponds to part (3) of Proposition 3.12. Finally, since the framings of the Pontrjagin-Thom arcs differ from the standard frame paths by the standard frame assignment f , the second term of Formula (3.6) corresponds to part (2) of Proposition 3.12. This completes the proof. \square

4. THE KHOVANOV HOMOTOPY TYPE OF WIDTH THREE KNOTS

It is immediate from Whitehead's theorem that if $\tilde{H}_i(X)$ is trivial for all $i \neq m$ then X is a Moore space; in particular, the homotopy type of X is determined by the homology of X in this case. This result can be extended to spaces with nontrivial homology in several gradings, if one also keeps track of the action of the Steenrod algebra. To determine the Khovanov homotopy types of links up to 11 crossings we will use such an extension due to Whitehead [Whi50] and Chang [Cha56], which we review here. (For further discussion along these lines, as well as the next larger case, see [Bau95, Section 11].)

Proposition 4.1. *Consider quivers of the form*

$$\begin{array}{ccccc} & & s & & \\ & \curvearrowright & & \curvearrowleft & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

where A , B and C are \mathbb{F}_2 -vector spaces, and $gf = 0$. Such a quiver uniquely decomposes as a direct sum of the following quivers:

$$\begin{array}{llll}
(S-1) & \mathbb{F}_2 & 0 & 0 \\
(S-2) & 0 & \mathbb{F}_2 & 0 \\
(S-3) & 0 & 0 & \mathbb{F}_2 \\
(P-1) & \mathbb{F}_2 \xrightarrow{\text{Id}} \mathbb{F}_2 & 0 & \\
(P-2) & 0 & \mathbb{F}_2 \xrightarrow{\text{Id}} \mathbb{F}_2 & \\
(X-1) & \mathbb{F}_2 \xrightarrow{\text{Id}} \mathbb{F}_2 & & \\
(X-2) & \mathbb{F}_2 \xrightarrow{\text{Id}} \mathbb{F}_2 & \mathbb{F}_2 & \\
(X-3) & \mathbb{F}_2 \xrightarrow{\text{Id}} \mathbb{F}_2 & \mathbb{F}_2 & \\
(X-4) & \mathbb{F}_2 \xrightarrow{\begin{pmatrix} \text{Id} \\ 0 \end{pmatrix}} \mathbb{F}_2 \oplus \mathbb{F}_2 \xrightarrow{(0, \text{Id})} \mathbb{F}_2. & &
\end{array}$$

Proof. We start with uniqueness. In such a decomposition, let s_i be the number of (S-i) summands, p_i be the number of (P-i) summands, and x_i be the number of (X-i) summands. Consider the following nine pieces of data:

- The dimensions of the \mathbb{F}_2 -vector spaces A , B and C , say d_1, d_2, d_3 , respectively;
- The ranks of the maps f and g , say r_f, r_g , respectively;
- The dimensions of the \mathbb{F}_2 -vector spaces $\text{im}(s)$, $\text{im}(s|_{\ker(f)})$, $\text{im}(g) \cap \text{im}(s)$ and $\text{im}(g) \cap \text{im}(s|_{\ker(f)})$, say r_1, r_2, r_3, r_4 , respectively.

We have

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ r_f \\ r_g \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 1 & . & . & 1 & . & 1 & 1 & 1 & 1 \\ . & 1 & . & 1 & 1 & . & 1 & 1 & 2 \\ . & . & 1 & . & 1 & 1 & 1 & 1 & 1 \\ . & . & . & 1 & . & . & 1 & . & 1 \\ . & . & . & . & 1 & . & . & 1 & 1 \\ . & . & . & . & . & 1 & 1 & 1 & 1 \\ . & . & . & . & . & 1 & . & 1 & . \\ . & . & . & . & . & . & . & 1 & 1 \\ . & . & . & . & . & . & . & 1 & . \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ p_1 \\ p_2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and therefore, the numbers s_i , p_i and x_i are determined as follows:

$$(4.1) \quad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ p_1 \\ p_2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & . & . & -1 & . & . & -1 & . & . \\ . & 1 & . & -1 & -1 & . & . & . & . \\ . & . & 1 & . & -1 & -1 & . & 1 & . \\ . & . & . & 1 & . & -1 & 1 & . & . \\ . & . & . & . & 1 & . & . & -1 & . \\ . & . & . & . & . & . & 1 & . & -1 \\ . & . & . & . & . & 1 & -1 & -1 & 1 \\ . & . & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & . & 1 & -1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ r_f \\ r_g \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}.$$

For existence of such a decomposition, we carry out a standard change-of-basis argument. Choose generators for A , B and C , and construct the following graph. There are three types of vertices, A -vertices, B -vertices and C -vertices, corresponding to generators of A , B and C respectively. There are three types of edges, f -edges, g -edges and s -edges, corresponding to the maps f , g and s as follows: for a an A -vertex and b a B -vertex, if b appears in $f(a)$ then there is an f -edge joining a and b ; the g -edges and s -edges are defined similarly.

We will do a change of basis, which will change the graph, so that in the final graph, each vertex is incident to at most one edge of each type. This will produce the required decomposition of the quiver.

We carry out the change of basis in the following sequence of steps. Each step accomplishes a specific simplification of the graph; it can be checked that the later steps do not undo the earlier simplifications.

- (1) We ensure that no two f -edges share a common vertex. Fix an f -edge joining an A -vertex a to a B -vertex b . Let $\{a_i\}$ be the other A -vertices that are f -adjacent to b and $\{b_j\}$ be the other B -vertices that are f -adjacent to a . Then change basis by replacing each a_i by $a_i + a$, and by replacing b with $b + \sum_j b_j$.
- (2) By the same procedure as Step (1), we ensure that no two g -edges share a common vertex.

Since $gf = 0$, this ensures that no B -vertex is adjacent to both an f -edge and a g -edge. Call an A -vertex an A_1 -vertex (resp. A_2 -vertex) if it is adjacent (resp. non-adjacent) to an f -edge; similarly, call a C -vertex a C_1 -vertex (resp. C_2 -vertex) if it is adjacent (resp. non-adjacent) to a g -vertex.

- (3) Next, we isolate the s -edges that connect A_2 -vertices to C_2 -vertices. Fix an s -edge joining an A_2 -vertex a to a C_2 -vertex c . If $\{a_i\}$ (resp. $\{c_j\}$) are the other A -vertices (resp. C -vertices) that are s -adjacent to c (resp. a), then change basis by replacing each a_i by $a_i + a$ and by replacing c with $c + \sum_j c_j$.
- (4) The next step is to isolate the s -edges that connect A_1 -vertices to C_2 -vertices. Once again, fix an s -edge joining an A_1 -vertex a to a C_2 -vertex c . Let $\{a_i\}$ (resp. $\{c_j\}$) be the other A -vertices (resp. C -vertices) that are s -adjacent to c (resp. a). Let b_i be the B -vertex that is f -adjacent to a_i (observe, each a_i is an A_1 -vertex), and let b be the B -vertex that is f -adjacent to a . Then change basis by replacing each a_i by $a_i + a$, by replacing each b_i by $b_i + b$ and by replacing c with $c + \sum_j c_j$.
- (5) Similarly, we can isolate the s -edges that connect A_2 -vertices to C_1 -vertices. As before, fix an s -edge joining an A_2 -vertex a to a C_1 -vertex c . Let $\{a_i\}$ (resp. $\{c_j\}$) be the other A -vertices (resp. C -vertices) that are s -adjacent to c (resp. a). Let b_j be the B -vertex that is g -adjacent to c_j and let b be the B -vertex that is g -adjacent to c . Then change basis by replacing each a_i by $a_i + a$, by replacing b with $b + \sum_j b_j$ and by replacing c with $c + \sum_j c_j$.

- (6) Finally, we have to isolate the s -edges that connect A_1 -vertices to C_1 -vertices. This can be accomplished by a combination of the previous two steps. \square

We are interested in stable spaces X satisfying the following conditions:

- The only torsion in $H^*(X; \mathbb{Z})$ is 2-torsion, and
- $\tilde{H}^i(X; \mathbb{F}_2) = 0$ if $i \neq 0, 1, 2$.

Then the quiver

$$\begin{array}{ccccc} & & \text{Sq}^2 & & \\ & \nearrow & & \searrow & \\ \tilde{H}^0(X; \mathbb{F}_2) & \xrightarrow{\text{Sq}^1} & \tilde{H}^1(X; \mathbb{F}_2) & \xrightarrow{\text{Sq}^1} & \tilde{H}^2(X; \mathbb{F}_2) \end{array}$$

is of the form described in [Proposition 4.1](#). In Examples 4.1–4.5, we will describe nine such spaces whose associated quivers are the nine irreducible ones of [Proposition 4.1](#).

Example 4.1. The associated quivers of S^0 , S^1 and S^2 are (S-1), (S-2) and (S-3), respectively. The associated quivers of $\Sigma^{-1}\mathbb{RP}^2$ and \mathbb{RP}^2 are (P-1) and (P-2), respectively.

Example 4.2. The space \mathbb{CP}^2 has cohomology

$$\begin{array}{ccc} \tilde{H}^4(\mathbb{CP}^2; \mathbb{Z}) & \mathbb{Z} & \tilde{H}^4(\mathbb{CP}^2; \mathbb{F}_2) & \mathbb{F}_2 \\ \tilde{H}^3(\mathbb{CP}^2; \mathbb{Z}) & 0 & \tilde{H}^3(\mathbb{CP}^2; \mathbb{F}_2) & 0 \\ \tilde{H}^2(\mathbb{CP}^2; \mathbb{Z}) & \mathbb{Z} & \tilde{H}^2(\mathbb{CP}^2; \mathbb{F}_2) & \mathbb{F}_2. \end{array} \quad \begin{array}{c} \uparrow \\ \text{Sq}^2 \end{array}$$

(The fact that Sq^2 has this form follows from the fact that for $x \in H^n$, $\text{Sq}^n(x) = x \cup x$.) Therefore, the stable space $X_1 := \Sigma^{-2}\mathbb{CP}^2$ has (X-1) as its associated quiver.

Example 4.3. The space $\mathbb{RP}^5/\mathbb{RP}^2$ has cohomology

$$\begin{array}{ccc} \tilde{H}^5(\mathbb{RP}^5/\mathbb{RP}^2; \mathbb{Z}) & \mathbb{Z} & \tilde{H}^5(\mathbb{RP}^5/\mathbb{RP}^2; \mathbb{F}_2) & \mathbb{F}_2 \\ \tilde{H}^4(\mathbb{RP}^5/\mathbb{RP}^2; \mathbb{Z}) & \mathbb{F}_2 & \tilde{H}^4(\mathbb{RP}^5/\mathbb{RP}^2; \mathbb{F}_2) & \mathbb{F}_2 \\ \tilde{H}^3(\mathbb{RP}^5/\mathbb{RP}^2; \mathbb{Z}) & 0 & \tilde{H}^3(\mathbb{RP}^5/\mathbb{RP}^2; \mathbb{F}_2) & \mathbb{F}_2. \end{array} \quad \begin{array}{c} \uparrow \\ \text{Sq}^1 \\ \uparrow \\ \text{Sq}^2 \end{array}$$

To see that Sq^2 has the stated form, consider the inclusion map $\mathbb{RP}^5/\mathbb{RP}^2 \rightarrow \mathbb{RP}^6/\mathbb{RP}^2$. The map $\text{Sq}^3: H^3(\mathbb{RP}^6/\mathbb{RP}^2) \rightarrow H^6(\mathbb{RP}^6/\mathbb{RP}^2)$ is an isomorphism (since it is just the cup

square). By the Adem relations, $Sq^3 = Sq^1 Sq^2$, so $Sq^2: H^3(\mathbb{RP}^6/\mathbb{RP}^2) \rightarrow H^5(\mathbb{RP}^6/\mathbb{RP}^2)$ is nontrivial. So, the corresponding statement for $\mathbb{RP}^5/\mathbb{RP}^2$ follows from naturality. Therefore, the stable space $X_2 := \Sigma^{-3}(\mathbb{RP}^5/\mathbb{RP}^2)$ has (X-2) as its associated quiver.

Example 4.4. The space $\mathbb{RP}^4/\mathbb{RP}^1$ has cohomology

$$\begin{array}{ccc}
 \tilde{H}^4(\mathbb{RP}^4/\mathbb{RP}^1; \mathbb{Z}) & \mathbb{F}_2 & \tilde{H}^4(\mathbb{RP}^4/\mathbb{RP}^1; \mathbb{F}_2) \\
 \tilde{H}^3(\mathbb{RP}^4/\mathbb{RP}^1; \mathbb{Z}) & 0 & \tilde{H}^3(\mathbb{RP}^4/\mathbb{RP}^1; \mathbb{F}_2) \\
 \tilde{H}^2(\mathbb{RP}^4/\mathbb{RP}^1; \mathbb{Z}) & \mathbb{Z} & \tilde{H}^2(\mathbb{RP}^4/\mathbb{RP}^1; \mathbb{F}_2)
 \end{array}
 \begin{array}{c}
 \mathbb{F}_2 \\
 \uparrow Sq^1 \\
 \mathbb{F}_2 \\
 \uparrow Sq^2 \\
 \mathbb{F}_2
 \end{array}$$

(The answer for Sq^2 again follows from the fact that it is the cup square.) Therefore, the stable space $X_3 := \Sigma^{-2}(\mathbb{RP}^4/\mathbb{RP}^1)$ has (X-3) as its associated quiver.

Example 4.5. The space $\mathbb{RP}^2 \wedge \mathbb{RP}^2$ has cohomology

$$\begin{array}{ccc}
 \tilde{H}^4(\mathbb{RP}^2 \wedge \mathbb{RP}^2; \mathbb{Z}) & \mathbb{F}_2 & \tilde{H}^4(\mathbb{RP}^2 \wedge \mathbb{RP}^2; \mathbb{F}_2) \\
 \tilde{H}^3(\mathbb{RP}^2 \wedge \mathbb{RP}^2; \mathbb{Z}) & \mathbb{F}_2 & \tilde{H}^3(\mathbb{RP}^2 \wedge \mathbb{RP}^2; \mathbb{F}_2) \\
 \tilde{H}^2(\mathbb{RP}^2 \wedge \mathbb{RP}^2; \mathbb{Z}) & 0 & \tilde{H}^2(\mathbb{RP}^2 \wedge \mathbb{RP}^2; \mathbb{F}_2)
 \end{array}
 \begin{array}{c}
 \mathbb{F}_2 \\
 \swarrow Sq^1 \quad \searrow Sq^1 \\
 \mathbb{F}_2 \oplus \mathbb{F}_2 \\
 \swarrow Sq^1 \quad \searrow Sq^1 \\
 \mathbb{F}_2
 \end{array}$$

(The answer for Sq^2 follows from the product formula: $Sq^2(a \wedge b) = a \wedge Sq^2(b) + Sq^1(a) \wedge Sq^1(b) + Sq^2(a) \wedge b$.) Therefore, the quiver associated to the stable space $X_4 := \Sigma^{-2}(\mathbb{RP}^2 \wedge \mathbb{RP}^2)$ is isomorphic to (X-4).

The following is a classification theorem from [Bau95, Theorems 11.2 and 11.7].

Proposition 4.2. *Let X be a simply connected CW complex such that:*

- *The only torsion in the cohomology of X is 2-torsion.*
- *There exists m sufficiently large so that the reduced cohomology $\tilde{H}^i(X; \mathbb{F}_2)$ is trivial for $i \neq m, m+1, m+2$.*

Then the homotopy type of X is determined by the isomorphism class of the quiver

$$\begin{array}{ccccc}
 & & Sq^2 & & \\
 & \curvearrowright & & \curvearrowleft & \\
 H^m(X; \mathbb{F}_2) & \xrightarrow{Sq^1} & H^{m+1}(X; \mathbb{F}_2) & \xrightarrow{Sq^1} & H^{m+2}(X; \mathbb{F}_2)
 \end{array}$$

as follows: Decompose the quiver as in [Proposition 4.1](#); let s_i be the number of (S-i) summands, $1 \leq i \leq 3$; let p_i be the number of (P-i) summands, $1 \leq i \leq 2$; and let x_i be the number of (X-i) summands, $1 \leq i \leq 4$. Then X is homotopy equivalent to

$$Y := \left(\bigvee_{i=1}^3 \bigvee_{j=1}^{s_i} S^{m+i-1} \right) \vee \left(\bigvee_{i=1}^2 \bigvee_{j=1}^{p_i} \Sigma^{m+i-2} \mathbb{RP}^2 \right) \vee \left(\bigvee_{i=1}^4 \bigvee_{j=1}^{x_i} \Sigma^m X_i \right).$$

In light of [Corollary 4.4](#), the following seems a natural link invariant.

Definition 4.3. For any link L , the function $St(L): \mathbb{Z}^2 \rightarrow \mathbb{N}^4$ is defined as follows: Fix $(i, j) \in \mathbb{Z}^2$; for $k \in \{i, i+1\}$, let $Sq_{(k)}^1$ denote the map $Sq^1: Kh^{k,j}(L) \rightarrow Kh^{k+1,j}(L)$.

Let r_1 be the rank of the map $Sq^2: Kh^{i,j}(L) \rightarrow Kh^{i+2,j}(L)$; let r_2 be the rank of the map $Sq^2|_{\ker Sq_{(i)}^1}$; let r_3 be the dimension of the \mathbb{F}_2 -vector space $\text{im } Sq_{(i+1)}^1 \cap \text{im } Sq^2$; and let r_4 be the dimension of the \mathbb{F}_2 -vector space $\text{im } Sq_{(i+1)}^1 \cap \text{im}(Sq^2|_{\ker Sq_{(i)}^1})$. Then,

$$St(i, j) := (r_2 - r_4, r_1 - r_2 - r_3 + r_4, r_4, r_3 - r_4).$$

Corollary 4.4. Suppose that the Khovanov homology $Kh_{\mathbb{Z}}(L)$ satisfies the following properties:

- $Kh_{\mathbb{Z}}^{i,j}(L)$ lies on three adjacent diagonals, say $2i - j = \sigma, \sigma + 2, \sigma + 4$.
- There is no torsion other than 2-torsion.
- There is no torsion on the diagonal $2i - j = \sigma$.

Then the homotopy types of the stable spaces $\mathcal{X}_{Kh}^j(L)$ are determined by $Kh_{\mathbb{Z}}(L)$ and $St(L)$ as follows: Fix j ; let $i = \frac{j+\sigma}{2}$; let $St(i, j) = (x_1, x_2, x_3, x_4)$; then $\mathcal{X}_{Kh}^j(L)$ is stably homotopy equivalent to the wedge sum of

$$\left(\bigvee_{x_1} \Sigma^{i-2} \mathbb{CP}^2 \right) \vee \left(\bigvee_{x_2} \Sigma^{i-3} (\mathbb{RP}^5 / \mathbb{RP}^2) \right) \vee \left(\bigvee_{x_3} \Sigma^{i-2} (\mathbb{RP}^4 / \mathbb{RP}^1) \right) \vee \left(\bigvee_{x_4} \Sigma^{i-2} (\mathbb{RP}^2 \wedge \mathbb{RP}^2) \right)$$

and a wedge of Moore spaces. In particular, $\mathcal{X}_{Kh}^j(L)$ is a wedge of Moore spaces if and only if $x_1 = x_2 = x_3 = x_4 = 0$.

Proof. The first part is immediate from [Proposition 4.2](#). To wit, if one decomposes the quiver

$$\begin{array}{ccccc} & & Sq^2 & & \\ & \curvearrowright & & \curvearrowleft & \\ Kh_{\mathbb{F}_2}^{i,j} & \xrightarrow{Sq^1} & Kh_{\mathbb{F}_2}^{i+1,j} & \xrightarrow{Sq^1} & Kh_{\mathbb{F}_2}^{i+2,j} \end{array}$$

as a direct sum of the nine quivers of [Proposition 4.1](#), Equation (4.1) implies that the number of (X-i) summands will be x_i .

The ‘if’ direction of the second part follows from the first part. For the ‘only if’ direction, observe that the rank of $Sq^2: Kh_{\mathbb{F}_2}^{i,j} \rightarrow Kh_{\mathbb{F}_2}^{i+2,j}$ is $x_1 + x_2 + x_3 + x_4$; therefore, if $\mathcal{X}_{Kh}^j(L)$ is a wedge of Moore spaces, $Sq^2 = 0$, and hence $x_1 = x_2 = x_3 = x_4 = 0$. \square

and from Table 1, we know the function $St(K)$:

$$St(-4, -9) = (0, 0, 0, 1)$$

$$St(-6, -13) = (0, 0, 1, 0)$$

$$St(-7, -15) = (0, 1, 0, 0).$$

Therefore, via Corollary 4.4, we can compute Khovanov homotopy types:

$$\mathcal{X}_{Kh}^{-3}(K) \sim S^0$$

$$\mathcal{X}_{Kh}^{-5}(K) \sim S^0$$

$$\mathcal{X}_{Kh}^{-7}(K) \sim \Sigma^{-3}(S^0 \vee S^1)$$

$$\mathcal{X}_{Kh}^{-9}(K) \sim \Sigma^{-6}(\mathbb{RP}^2 \wedge \mathbb{RP}^2)$$

$$\mathcal{X}_{Kh}^{-11}(K) \sim \Sigma^{-5}(S^0 \vee S^1 \vee S^1 \vee S^2)$$

$$\mathcal{X}_{Kh}^{-13}(K) \sim \Sigma^{-8}(\mathbb{RP}^4/\mathbb{RP}^1 \vee \Sigma\mathbb{RP}^2)$$

$$\mathcal{X}_{Kh}^{-15}(K) \sim \Sigma^{-10}(\mathbb{RP}^5/\mathbb{RP}^2 \vee S^4)$$

$$\mathcal{X}_{Kh}^{-17}(K) \sim \Sigma^{-8}(S^0 \vee S^1)$$

$$\mathcal{X}_{Kh}^{-19}(K) \sim \Sigma^{-10}\mathbb{RP}^2$$

$$\mathcal{X}_{Kh}^{-21}(K) \sim \Sigma^{-9}S^0.$$

Example 5.2. The Kinoshita-Terasaka knot $K_1 = K11n42$ and its Conway mutant $K_2 = K11n34$ have identical Khovanov homology. From Table 1, we see that $St(K_1) = St(K_2)$. Therefore, by Corollary 4.4, they have the same Khovanov homotopy type.

The Kinoshita-Terasaka knot and its Conway mutant is an example of a pair of links that are not distinguished by their Khovanov homologies. It is natural to ask:

Question 5.1. Does there exist a pair of links L_1 and L_2 with $Kh_{\mathbb{Z}}(L_1) = Kh_{\mathbb{Z}}(L_2)$, but $\mathcal{X}_{Kh}(L_1) \not\sim \mathcal{X}_{Kh}(L_2)$?

The following example is a partial answer.

Example 5.3. The links $L_1 = L11n383$ and $L_2 = L11n393$ have isomorphic Khovanov homology in quantum grading (-3) : $Kh_{\mathbb{Z}}^{-2,-3} = \mathbb{Z}^3$, $Kh_{\mathbb{Z}}^{-1,-3} = \mathbb{Z}^3 \oplus \mathbb{F}_2^4$, $Kh_{\mathbb{Z}}^{0,-3} = \mathbb{Z}^2$, [KAT]. However, $St(L_1)(-2, -3) = (0, 2, 0, 0)$ and $St(L_2)(-2, -3) = (0, 1, 0, 0)$ (Table 1); therefore, $\mathcal{X}_{Kh}^{-3}(L_1)$ is not stably homotopy equivalent to $\mathcal{X}_{Kh}^{-3}(L_2)$.

We conclude with an observation and a question. Since all prime links up to 11 crossings satisfy the conditions of Corollary 4.4, their Khovanov homotopy types are wedges of various suspensions of S^0 , \mathbb{RP}^2 , \mathbb{CP}^2 , $\mathbb{RP}^5/\mathbb{RP}^2$, $\mathbb{RP}^4/\mathbb{RP}^1$ and $\mathbb{RP}^2 \wedge \mathbb{RP}^2$; and this wedge sum decomposition is unique since it is determined by the Khovanov homology $Kh_{\mathbb{Z}}$ and the function St . Example 5.1 already exhibits all but one of these summands; it does not have a \mathbb{CP}^2 summand. A careful look at Table 1 reveals that neither does any other link up to 11 crossings. The conspicuous absence of \mathbb{CP}^2 naturally leads to the following question.

Question 5.2. Does there exist a link L for which $\mathcal{X}_{Kh}^j(L)$ contains $\Sigma^m \mathbb{CP}^2$ in some⁽ⁱⁱ⁾ wedge sum decomposition, for some j, m ?

⁽ⁱⁱ⁾Wedge sum decompositions are in general not unique.

TABLE 1

L	$St(L)$
8 ₁₉	$(2, 11) \mapsto (0, 1, 0, 0)$
9 ₄₂	$(-2, -1) \mapsto (0, 1, 0, 0)$
10 ₁₂₄	$(2, 13) \mapsto (0, 1, 0, 0), (5, 19) \mapsto (0, 0, 1, 0)$
10 ₁₂₈	$(2, 11) \mapsto (0, 1, 0, 0)$
10 ₁₃₂	$(-5, -9) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 1, 0, 0)$
10 ₁₃₆	$(-2, -1) \mapsto (0, 1, 0, 0)$
10 ₁₃₉	$(2, 13) \mapsto (0, 1, 0, 0), (5, 19) \mapsto (0, 0, 1, 0)$
10 ₁₄₅	$(-4, -9) \mapsto (0, 0, 0, 1), (-6, -13) \mapsto (0, 0, 1, 0), (-7, -15) \mapsto (0, 1, 0, 0)$
10 ₁₅₂	$(-7, -19) \mapsto (0, 1, 0, 0), (-4, -13) \mapsto (0, 0, 1, 0)$
10 ₁₅₃	$(0, 1) \mapsto (0, 1, 0, 0), (1, 3) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 1, 0, 0)$
10 ₁₅₄	$(5, 17) \mapsto (0, 0, 1, 0), (2, 11) \mapsto (0, 1, 0, 0)$
10 ₁₆₁	$(-4, -11) \mapsto (0, 0, 1, 0), (-7, -17) \mapsto (0, 1, 0, 0)$
K11n6	$(0, 1) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0), (-4, -7) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 1, 0, 0)$
K11n9	$(3, 13) \mapsto (0, 1, 0, 0), (5, 17) \mapsto (0, 0, 1, 0), (4, 15) \mapsto (0, 0, 1, 0), (2, 11) \mapsto (0, 1, 0, 0), (0, 7) \mapsto (0, 1, 0, 0), (1, 9) \mapsto (0, 0, 1, 0)$
K11n12	$(2, 7) \mapsto (0, 1, 0, 0), (0, 3) \mapsto (0, 0, 1, 0), (3, 9) \mapsto (0, 0, 1, 0)$
K11n19	$(0, -1) \mapsto (0, 0, 1, 0), (-3, -7) \mapsto (0, 0, 0, 1)$
K11n20	$(0, 1) \mapsto (0, 0, 1, 0)$
K11n24	$(-2, -1) \mapsto (0, 1, 0, 0)$
K11n27	$(2, 11) \mapsto (0, 1, 0, 0)$
K11n31	$(2, 9) \mapsto (0, 0, 0, 1), (3, 11) \mapsto (0, 1, 0, 0), (4, 13) \mapsto (0, 1, 1, 0), (5, 15) \mapsto (0, 0, 1, 0), (0, 5) \mapsto (0, 1, 0, 0), (1, 7) \mapsto (0, 0, 1, 0)$
K11n34	$(0, 1) \mapsto (0, 1, 1, 0), (1, 3) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 0, 1, 0), (-1, -1) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto (0, 1, 0, 0), (-3, -5) \mapsto (0, 1, 1, 0)$
K11n38	$(-1, 1) \mapsto (0, 1, 0, 0), (0, 3) \mapsto (0, 0, 1, 0), (-4, -5) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 0, 1, 0)$
K11n39	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 0, 1, 0)$
K11n42	$(0, 1) \mapsto (0, 1, 1, 0), (1, 3) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 0, 1, 0), (-1, -1) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto (0, 1, 0, 0), (-3, -5) \mapsto (0, 1, 1, 0)$
K11n45	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 0, 1, 0)$
K11n49	$(0, 3) \mapsto (0, 0, 1, 0), (-3, -3) \mapsto (0, 0, 1, 0), (-4, -5) \mapsto (0, 1, 0, 0), (-1, 1) \mapsto (0, 1, 0, 0)$
K11n57	$(4, 15) \mapsto (0, 0, 1, 0), (2, 11) \mapsto (0, 1, 0, 0), (1, 9) \mapsto (0, 0, 1, 0), (0, 7) \mapsto (0, 1, 0, 0), (3, 13) \mapsto (0, 1, 0, 0)$
K11n61	$(-1, 3) \mapsto (0, 1, 0, 0), (0, 5) \mapsto (0, 0, 1, 0), (2, 9) \mapsto (0, 1, 0, 0)$
K11n67	$(2, 7) \mapsto (0, 0, 1, 0), (1, 5) \mapsto (0, 1, 0, 0), (-1, 1) \mapsto (0, 0, 1, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
K11n70	$(-2, 1) \mapsto (0, 1, 0, 0), (0, 5) \mapsto (0, 1, 0, 0), (1, 7) \mapsto (0, 0, 1, 0)$
K11n73	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 0, 1, 0)$
K11n74	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 0, 1, 0)$
K11n77	$(2, 13) \mapsto (0, 1, 0, 0), (5, 19) \mapsto (0, 0, 1, 0)$
K11n79	$(-2, -1) \mapsto (0, 1, 0, 0)$
K11n80	$(-3, -7) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 0, 0), (-1, -3) \mapsto (0, 1, 0, 0), (0, -1) \mapsto (0, 0, 1, 0)$
K11n81	$(2, 11) \mapsto (0, 1, 0, 0)$
K11n88	$(2, 11) \mapsto (0, 1, 0, 0)$
K11n92	$(0, 1) \mapsto (0, 0, 1, 0)$
K11n96	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 1, 0)$
K11n97	$(2, 5) \mapsto (0, 0, 1, 0), (1, 3) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 0, 1, 0)$
K11n102	$(-5, -9) \mapsto (0, 0, 1, 0), (-6, -11) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 1, 0, 0)$
K11n104	$(4, 15) \mapsto (0, 0, 1, 0), (2, 11) \mapsto (0, 1, 0, 0), (1, 9) \mapsto (0, 0, 1, 0), (0, 7) \mapsto (0, 1, 0, 0), (3, 13) \mapsto (0, 1, 0, 0)$
K11n111	$(-1, 3) \mapsto (0, 0, 1, 0), (-2, 1) \mapsto (0, 1, 0, 0), (2, 9) \mapsto (0, 0, 1, 0), (1, 7) \mapsto (0, 1, 0, 0)$
K11n116	$(0, 1) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0), (-4, -7) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 1, 0, 0)$
K11n126	$(2, 11) \mapsto (0, 1, 0, 0)$
K11n133	$(-1, 3) \mapsto (0, 1, 0, 0), (0, 5) \mapsto (0, 0, 1, 0), (2, 9) \mapsto (0, 1, 0, 0)$
K11n135	$(4, 13) \mapsto (0, 0, 1, 0), (0, 5) \mapsto (0, 1, 0, 0), (3, 11) \mapsto (0, 1, 0, 0), (1, 7) \mapsto (0, 0, 1, 0)$

K11n138	$(-2, -1) \mapsto (0, 1, 0, 0)$
K11n143	$(2, 7) \mapsto (0, 0, 1, 0), (1, 5) \mapsto (0, 1, 0, 0), (-1, 1) \mapsto (0, 0, 1, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
K11n145	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 0, 1, 0)$
K11n151	$(-1, 3) \mapsto (0, 0, 1, 0), (-2, 1) \mapsto (0, 1, 0, 0), (2, 9) \mapsto (0, 0, 1, 0), (1, 7) \mapsto (0, 1, 0, 0)$
K11n152	$(-1, 3) \mapsto (0, 0, 1, 0), (-2, 1) \mapsto (0, 1, 0, 0), (2, 9) \mapsto (0, 0, 1, 0), (1, 7) \mapsto (0, 1, 0, 0)$
K11n183	$(5, 17) \mapsto (0, 0, 1, 0), (2, 11) \mapsto (0, 1, 0, 0)$
L6n1	$(0, 3) \mapsto (0, 0, 1, 0)$
L7n1	$(-4, -10) \mapsto (0, 0, 1, 0)$
L8n2	$(-2, -2) \mapsto (0, 1, 0, 0)$
L8n3	$(-4, -11) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 0, 1, 0)$
L8n6	$(-6, -13) \mapsto (0, 1, 0, 0)$
L8n7	$(0, 4) \mapsto (0, 0, 1, 0)$
L8n8	$(0, 2) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L9n1	$(-4, -10) \mapsto (0, 0, 1, 0)$
L9n3	$(-2, -4) \mapsto (0, 1, 0, 0), (-5, -10) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 0, 1, 0)$
L9n4	$(-6, -16) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-7, -18) \mapsto (0, 1, 0, 0)$
L9n9	$(-7, -16) \mapsto (0, 1, 0, 0), (-4, -10) \mapsto (0, 0, 1, 0), (-6, -14) \mapsto (0, 0, 1, 0)$
L9n12	$(1, 6) \mapsto (0, 0, 1, 0), (-2, 0) \mapsto (0, 1, 0, 0)$
L9n15	$(-4, -12) \mapsto (0, 0, 1, 0)$
L9n18	$(-4, -12) \mapsto (0, 0, 1, 0)$
L9n21	$(0, 1) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 1, 0, 0)$
L9n22	$(-2, -3) \mapsto (0, 1, 0, 0)$
L9n23	$(-2, -5) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 0, 1, 0)$
L9n25	$(-2, -3) \mapsto (0, 1, 0, 0)$
L9n26	$(-2, -3) \mapsto (0, 1, 0, 0)$
L9n27	$(0, 1) \mapsto (0, 0, 1, 0)$
L10n1	$(-2, -4) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 0, 1, 0), (-1, -2) \mapsto (0, 0, 1, 0)$
L10n3	$(-2, -2) \mapsto (0, 1, 0, 0)$
L10n5	$(1, 6) \mapsto (0, 0, 1, 0), (-2, 0) \mapsto (0, 1, 0, 0), (0, 4) \mapsto (0, 1, 0, 0)$
L10n8	$(-2, -2) \mapsto (0, 1, 0, 0)$
L10n9	$(2, 6) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 0, 1, 0), (0, 2) \mapsto (0, 0, 1, 0), (1, 4) \mapsto (0, 1, 0, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L10n10	$(-2, -6) \mapsto (0, 0, 1, 0), (-5, -12) \mapsto (0, 0, 1, 0), (-4, -10) \mapsto (0, 0, 1, 0), (-3, -8) \mapsto (0, 1, 0, 0), (-6, -14) \mapsto (0, 1, 0, 0)$
L10n13	$(-4, -10) \mapsto (0, 0, 1, 0)$
L10n14	$(0, 0) \mapsto (0, 0, 1, 0), (-3, -6) \mapsto (0, 0, 1, 0), (-1, -2) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 0, 0)$
L10n18	$(-3, -4) \mapsto (0, 1, 0, 0), (0, 2) \mapsto (0, 1, 1, 0), (1, 4) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 0, 1, 0)$
L10n23	$(2, 10) \mapsto (0, 1, 0, 0)$
L10n24	$(0, 2) \mapsto (0, 0, 1, 0)$
L10n25	$(-4, -6) \mapsto (0, 1, 0, 0), (-3, -4) \mapsto (0, 0, 1, 0), (0, 2) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 1, 0, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L10n28	$(-2, -4) \mapsto (0, 0, 1, 0), (-3, -6) \mapsto (0, 1, 0, 0), (-6, -12) \mapsto (0, 1, 0, 0), (-5, -10) \mapsto (0, 0, 1, 0)$
L10n32	$(-4, -6) \mapsto (0, 1, 0, 0), (-3, -4) \mapsto (0, 0, 1, 0), (0, 2) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 1, 0, 0)$
L10n36	$(-3, -4) \mapsto (0, 1, 0, 0), (0, 2) \mapsto (0, 1, 0, 0), (1, 4) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 0, 1, 0)$
L10n37	$(1, 6) \mapsto (0, 0, 1, 0), (-2, 0) \mapsto (0, 1, 0, 0)$
L10n39	$(2, 10) \mapsto (0, 1, 0, 0)$
L10n42	$(0, 6) \mapsto (0, 1, 0, 0), (2, 10) \mapsto (0, 1, 0, 0)$
L10n45	$(0, 0) \mapsto (0, 0, 1, 0)$
L10n54	$(2, 10) \mapsto (0, 1, 0, 0)$
L10n56	$(0, 0) \mapsto (0, 0, 1, 0)$
L10n59	$(-3, -4) \mapsto (0, 1, 0, 0), (0, 2) \mapsto (0, 1, 0, 0), (1, 4) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 0, 1, 0)$
L10n60	$(1, 6) \mapsto (0, 0, 1, 0), (-2, 0) \mapsto (0, 1, 0, 0)$
L10n62	$(2, 10) \mapsto (0, 1, 0, 0)$
L10n66	$(0, 1) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0), (-4, -7) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 1, 0, 0)$
L10n67	$(-2, -3) \mapsto (0, 1, 0, 0)$

L10n68	$(-4, -11) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 0, 1, 0)$
L10n70	$(-2, -5) \mapsto (0, 1, 0, 0), (-4, -9) \mapsto (0, 0, 1, 0), (-6, -13) \mapsto (0, 1, 0, 0), (-5, -11) \mapsto (0, 1, 0, 0)$
L10n72	$(2, 7) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 0, 1, 0), (-1, 1) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L10n74	$(-4, -11) \mapsto (0, 0, 1, 0), (-8, -19) \mapsto (0, 0, 1, 0), (-5, -13) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 1, 0, 0)$
L10n77	$(-6, -17) \mapsto (0, 0, 1, 0), (-7, -19) \mapsto (0, 1, 0, 0), (-8, -21) \mapsto (0, 0, 1, 0), (-4, -13) \mapsto (0, 0, 1, 0)$
L10n82	$(-6, -11) \mapsto (0, 1, 0, 0), (-3, -5) \mapsto (0, 0, 1, 0)$
L10n83	$(-4, -9) \mapsto (0, 0, 1, 0)$
L10n84	$(-4, -11) \mapsto (0, 0, 1, 0), (-8, -19) \mapsto (0, 1, 0, 0), (-6, -15) \mapsto (0, 0, 1, 0), (-7, -17) \mapsto (0, 1, 0, 0)$
L10n87	$(-2, 1) \mapsto (0, 1, 0, 0), (1, 7) \mapsto (0, 0, 1, 0)$
L10n88	$(0, 1) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 1, 0, 0)$
L10n91	$(-6, -13) \mapsto (0, 1, 0, 0)$
L10n93	$(-4, -13) \mapsto (0, 0, 1, 0)$
L10n97	$(-2, -4) \mapsto (0, 0, 1, 0), (0, 0) \mapsto (0, 0, 1, 0), (-3, -6) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 0, 0)$
L10n98	$(0, 2) \mapsto (0, 1, 0, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L10n101	$(2, 8) \mapsto (0, 0, 1, 0), (4, 12) \mapsto (0, 0, 1, 0), (1, 6) \mapsto (0, 1, 0, 0)$
L10n102	$(-4, -10) \mapsto (0, 0, 1, 0), (-6, -14) \mapsto (0, 3, 0, 0)$
L10n103	$(-4, -8) \mapsto (0, 0, 1, 0)$
L10n104	$(-6, -12) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 0, 0)$
L10n106	$(-2, -4) \mapsto (0, 0, 1, 0), (-4, -8) \mapsto (0, 0, 1, 0)$
L10n107	$(0, 2) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L10n108	$(-2, -6) \mapsto (0, 0, 1, 0), (-4, -10) \mapsto (0, 0, 2, 0), (-6, -14) \mapsto (0, 0, 1, 0)$
L10n111	$(-4, -6) \mapsto (0, 1, 0, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L10n112	$(0, 5) \mapsto (0, 0, 1, 0)$
L10n113	$(0, 3) \mapsto (0, 0, 4, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L11n1	$(-2, -6) \mapsto (0, 0, 1, 0), (-5, -12) \mapsto (0, 0, 1, 0), (-4, -10) \mapsto (0, 0, 1, 0), (-3, -8) \mapsto (0, 1, 0, 0), (-6, -14) \mapsto (0, 1, 0, 0)$
L11n5	$(-4, -6) \mapsto (0, 1, 0, 0), (-3, -4) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 1, 0, 0), (0, 2) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L11n6	$(0, 2) \mapsto (0, 0, 1, 0)$
L11n8	$(-2, -4) \mapsto (0, 1, 0, 0), (-5, -10) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 0, 1, 0)$
L11n9	$(0, 0) \mapsto (0, 0, 1, 0), (-2, -4) \mapsto (0, 1, 1, 0), (-1, -2) \mapsto (0, 0, 1, 0), (-3, -6) \mapsto (0, 1, 0, 0), (-5, -10) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 0, 1, 0)$
L11n10	$(-6, -16) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-7, -18) \mapsto (0, 1, 0, 0)$
L11n12	$(-7, -16) \mapsto (0, 1, 0, 0), (-2, -6) \mapsto (0, 1, 0, 0), (-5, -12) \mapsto (0, 1, 0, 0), (-4, -10) \mapsto (0, 0, 1, 0), (-6, -14) \mapsto (0, 0, 1, 0)$
L11n13	$(-4, -10) \mapsto (0, 0, 1, 0)$
L11n15	$(-6, -12) \mapsto (0, 0, 1, 0), (-7, -14) \mapsto (0, 1, 0, 0), (-2, -4) \mapsto (0, 1, 0, 0), (-3, -6) \mapsto (0, 0, 1, 0), (-5, -10) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 1, 0)$
L11n16	$(-9, -22) \mapsto (0, 1, 0, 0), (-5, -14) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-7, -18) \mapsto (0, 1, 0, 0), (-8, -20) \mapsto (0, 0, 1, 0), (-6, -16) \mapsto (0, 1, 1, 0)$
L11n19	$(-9, -22) \mapsto (0, 1, 0, 0), (-5, -14) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-7, -18) \mapsto (0, 1, 0, 0), (-8, -20) \mapsto (0, 0, 1, 0), (-6, -16) \mapsto (0, 1, 1, 0)$
L11n22	$(-2, -4) \mapsto (0, 1, 0, 0), (-1, -2) \mapsto (0, 0, 1, 0), (-4, -8) \mapsto (0, 0, 1, 0)$
L11n24	$(2, 6) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 0, 1, 0), (1, 4) \mapsto (0, 1, 0, 0), (-2, -2) \mapsto (0, 2, 0, 0)$
L11n27	$(-2, -4) \mapsto (0, 1, 0, 0), (-5, -10) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 0, 1, 0)$
L11n28	$(-7, -14) \mapsto (0, 1, 0, 0), (-3, -6) \mapsto (0, 0, 1, 0), (-6, -12) \mapsto (0, 0, 1, 0), (-4, -8) \mapsto (0, 1, 0, 0)$
L11n30	$(-6, -12) \mapsto (0, 0, 1, 0), (-7, -14) \mapsto (0, 1, 0, 0), (-2, -4) \mapsto (0, 1, 0, 0), (-3, -6) \mapsto (0, 0, 1, 0), (-5, -10) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 1, 0)$
L11n33	$(-3, -4) \mapsto (0, 1, 0, 0), (2, 6) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 0, 1, 0), (1, 4) \mapsto (0, 1, 1, 0), (-2, -2) \mapsto (0, 1, 1, 0), (0, 2) \mapsto (0, 1, 1, 0)$
L11n38	$(-4, -10) \mapsto (0, 0, 1, 0)$
L11n39	$(0, 0) \mapsto (0, 0, 1, 0), (-2, -4) \mapsto (0, 1, 0, 0), (-1, -2) \mapsto (0, 1, 1, 0), (-3, -6) \mapsto (0, 0, 1, 0), (-5, -10) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 1, 0)$
L11n41	$(-4, -10) \mapsto (0, 0, 1, 0)$

L11n44	$(-4, -14) \mapsto (0, 0, 1, 0), (-6, -18) \mapsto (0, 0, 1, 0), (-9, -24) \mapsto (0, 1, 0, 0), (-8, -22) \mapsto (0, 0, 1, 0),$ $(-7, -20) \mapsto (0, 1, 0, 0)$
L11n48	$(-7, -16) \mapsto (0, 1, 0, 0), (-2, -6) \mapsto (0, 1, 0, 0), (-5, -12) \mapsto (0, 1, 0, 0), (-4, -10) \mapsto (0, 0, 1, 0), (-6, -14) \mapsto$ $(0, 0, 1, 0)$
L11n52	$(-2, -2) \mapsto (0, 1, 0, 0)$
L11n53	$(0, 2) \mapsto (0, 0, 1, 0)$
L11n54	$(-2, 0) \mapsto (0, 1, 0, 0), (2, 8) \mapsto (0, 1, 0, 0), (3, 10) \mapsto (0, 0, 1, 0), (-1, 2) \mapsto (0, 1, 0, 0), (1, 6) \mapsto (0, 0, 1, 0),$ $(0, 4) \mapsto (0, 1, 1, 0)$
L11n57	$(-3, -4) \mapsto (0, 1, 0, 0), (0, 2) \mapsto (0, 1, 0, 0), (1, 4) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 1, 1, 0)$
L11n59	$(1, 6) \mapsto (0, 0, 1, 0), (-2, 0) \mapsto (0, 1, 0, 0)$
L11n61	$(-2, -2) \mapsto (0, 1, 0, 0)$
L11n62	$(-4, -12) \mapsto (0, 0, 1, 0), (-7, -18) \mapsto (0, 1, 0, 0)$
L11n64	$(-7, -16) \mapsto (0, 1, 0, 0), (-4, -10) \mapsto (0, 1, 1, 0), (-3, -8) \mapsto (0, 0, 1, 0), (-6, -14) \mapsto (0, 0, 1, 0)$
L11n65	$(-9, -20) \mapsto (0, 1, 0, 0), (-8, -18) \mapsto (0, 0, 1, 0), (-5, -12) \mapsto (0, 0, 1, 0), (-6, -14) \mapsto (0, 1, 0, 0)$
L11n68	$(-6, -16) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-7, -18) \mapsto (0, 1, 0, 0), (-9, -22) \mapsto (0, 1, 0, 0),$ $(-8, -20) \mapsto (0, 0, 1, 0)$
L11n71	$(-7, -16) \mapsto (0, 1, 0, 0), (-5, -12) \mapsto (0, 0, 1, 0), (-4, -10) \mapsto (0, 0, 1, 0), (-8, -18) \mapsto (0, 0, 1, 0),$ $(-9, -20) \mapsto (0, 1, 0, 0), (-6, -14) \mapsto (0, 1, 1, 0)$
L11n74	$(1, 8) \mapsto (0, 0, 1, 0), (-2, 2) \mapsto (0, 1, 0, 0)$
L11n75	$(-2, 0) \mapsto (0, 1, 0, 0), (2, 8) \mapsto (0, 1, 0, 0), (3, 10) \mapsto (0, 0, 1, 0), (-1, 2) \mapsto (0, 1, 0, 0), (1, 6) \mapsto (0, 0, 1, 0),$ $(0, 4) \mapsto (0, 0, 1, 0)$
L11n87	$(-4, -8) \mapsto (0, 0, 1, 0), (-2, -4) \mapsto (0, 1, 0, 0), (0, 0) \mapsto (0, 0, 1, 0), (-1, -2) \mapsto (0, 0, 1, 0)$
L11n89	$(-6, -12) \mapsto (0, 1, 0, 0), (-2, -4) \mapsto (0, 1, 1, 0), (-1, -2) \mapsto (0, 0, 1, 0), (-3, -6) \mapsto (0, 1, 0, 0), (-5, -10) \mapsto$ $(0, 1, 1, 0), (-4, -8) \mapsto (0, 0, 1, 0)$
L11n91	$(-7, -14) \mapsto (0, 1, 0, 0), (-3, -6) \mapsto (0, 0, 1, 0), (-6, -12) \mapsto (0, 0, 1, 0), (-4, -8) \mapsto (0, 1, 0, 0)$
L11n94	$(-2, 0) \mapsto (0, 1, 0, 0), (2, 8) \mapsto (0, 1, 0, 0), (3, 10) \mapsto (0, 0, 1, 0), (-1, 2) \mapsto (0, 1, 0, 0), (1, 6) \mapsto (0, 0, 1, 0),$ $(0, 4) \mapsto (0, 1, 1, 0)$
L11n95	$(-6, -16) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-7, -18) \mapsto (0, 1, 0, 0), (-9, -22) \mapsto (0, 1, 0, 0),$ $(-8, -20) \mapsto (0, 0, 1, 0)$
L11n98	$(1, 8) \mapsto (0, 0, 1, 0), (-2, 2) \mapsto (0, 1, 0, 0)$
L11n99	$(-2, 0) \mapsto (0, 1, 0, 0), (2, 8) \mapsto (0, 1, 0, 0), (3, 10) \mapsto (0, 0, 1, 0), (-1, 2) \mapsto (0, 1, 0, 0), (1, 6) \mapsto (0, 0, 1, 0),$ $(0, 4) \mapsto (0, 1, 1, 0)$
L11n103	$(-9, -20) \mapsto (0, 1, 0, 0), (-8, -18) \mapsto (0, 0, 1, 0), (-5, -12) \mapsto (0, 0, 1, 0), (-6, -14) \mapsto (0, 1, 0, 0)$
L11n106	$(-7, -16) \mapsto (0, 1, 0, 0), (-2, -6) \mapsto (0, 0, 1, 0), (-5, -12) \mapsto (0, 0, 1, 0), (-4, -10) \mapsto (0, 0, 1, 0), (-3, -8) \mapsto$ $(0, 1, 0, 0), (-6, -14) \mapsto (0, 1, 1, 0)$
L11n108	$(-2, 0) \mapsto (0, 1, 0, 0), (2, 8) \mapsto (0, 1, 0, 0), (3, 10) \mapsto (0, 0, 1, 0), (-1, 2) \mapsto (0, 1, 0, 0), (1, 6) \mapsto (0, 0, 1, 0),$ $(0, 4) \mapsto (0, 0, 1, 0)$
L11n109	$(-9, -20) \mapsto (0, 1, 0, 0), (-8, -18) \mapsto (0, 0, 1, 0), (-5, -12) \mapsto (0, 0, 1, 0), (-4, -10) \mapsto (0, 0, 1, 0),$ $(-6, -14) \mapsto (0, 1, 0, 0)$
L11n111	$(-7, -16) \mapsto (0, 1, 0, 0), (-2, -6) \mapsto (0, 0, 1, 0), (-5, -12) \mapsto (0, 0, 1, 0), (-4, -10) \mapsto (0, 0, 1, 0), (-3, -8) \mapsto$ $(0, 1, 0, 0), (-6, -14) \mapsto (0, 1, 1, 0)$
L11n112	$(2, 8) \mapsto (0, 0, 1, 0), (-1, 2) \mapsto (0, 0, 1, 0), (1, 6) \mapsto (0, 1, 0, 0), (-2, 0) \mapsto (0, 1, 0, 0)$
L11n115	$(-2, -2) \mapsto (0, 1, 0, 0)$
L11n120	$(-3, -4) \mapsto (0, 1, 1, 0), (-1, 0) \mapsto (0, 1, 0, 0), (1, 4) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 1, 1, 0), (-4, -6) \mapsto$ $(0, 1, 0, 0), (0, 2) \mapsto (0, 1, 1, 0)$
L11n121	$(1, 6) \mapsto (0, 0, 1, 0), (-2, 0) \mapsto (0, 1, 0, 0)$
L11n122	$(0, 2) \mapsto (0, 1, 0, 0), (1, 4) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L11n124	$(0, 2) \mapsto (0, 0, 1, 0)$
L11n125	$(-4, -8) \mapsto (0, 1, 0, 0), (0, 0) \mapsto (0, 0, 1, 0), (-3, -6) \mapsto (0, 0, 1, 0), (-1, -2) \mapsto (0, 1, 0, 0)$
L11n127	$(-4, -6) \mapsto (0, 1, 0, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L11n132	$(-7, -16) \mapsto (0, 1, 0, 0), (-4, -10) \mapsto (0, 0, 1, 0)$
L11n133	$(-4, -14) \mapsto (0, 0, 1, 0), (-7, -20) \mapsto (0, 1, 0, 0)$
L11n139	$(3, 8) \mapsto (0, 0, 1, 0), (2, 6) \mapsto (0, 1, 0, 0), (0, 2) \mapsto (0, 0, 1, 0)$
L11n141	$(-4, -12) \mapsto (0, 0, 1, 0)$
L11n148	$(-4, -6) \mapsto (0, 1, 0, 0), (-1, 0) \mapsto (0, 0, 1, 0)$

L11n159	$(0, 6) \mapsto (0, 1, 0, 0), (2, 10) \mapsto (0, 1, 0, 0)$
L11n162	$(-4, -6) \mapsto (0, 1, 0, 0), (-1, 0) \mapsto (0, 0, 1, 0)$
L11n165	$(-6, -16) \mapsto (0, 1, 0, 0), (-5, -14) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-9, -22) \mapsto (0, 1, 0, 0),$ $(-8, -20) \mapsto (0, 0, 1, 0)$
L11n166	$(1, 2) \mapsto (0, 0, 1, 0), (-2, -4) \mapsto (0, 0, 1, 0), (0, 0) \mapsto (0, 1, 0, 0), (-3, -6) \mapsto (0, 1, 0, 0)$
L11n169	$(-4, -14) \mapsto (0, 0, 1, 0), (-7, -20) \mapsto (0, 1, 0, 0)$
L11n176	$(-6, -16) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0)$
L11n180	$(-2, -4) \mapsto (0, 1, 0, 0), (0, 0) \mapsto (0, 0, 1, 0), (-1, -2) \mapsto (0, 0, 1, 0), (-4, -8) \mapsto (0, 0, 0, 1)$
L11n181	$(-4, -6) \mapsto (0, 1, 0, 0), (-1, 0) \mapsto (0, 0, 1, 0)$
L11n189	$(-4, -6) \mapsto (0, 1, 0, 0), (-1, 0) \mapsto (0, 0, 1, 0)$
L11n197	$(-4, -8) \mapsto (0, 0, 1, 0), (-2, -4) \mapsto (0, 1, 0, 0), (0, 0) \mapsto (0, 0, 1, 0), (-5, -10) \mapsto (0, 1, 0, 0), (-1, -2) \mapsto$ $(0, 0, 1, 0)$
L11n199	$(-6, -16) \mapsto (0, 1, 0, 0), (-5, -14) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-8, -20) \mapsto (0, 0, 1, 0)$
L11n204	$(-4, -14) \mapsto (0, 0, 1, 0), (-7, -20) \mapsto (0, 1, 0, 0)$
L11n218	$(3, 8) \mapsto (0, 0, 1, 0), (2, 6) \mapsto (0, 1, 0, 0), (0, 2) \mapsto (0, 0, 1, 0)$
L11n222	$(-4, -14) \mapsto (0, 0, 1, 0), (-7, -20) \mapsto (0, 1, 0, 0)$
L11n224	$(-4, -12) \mapsto (0, 0, 1, 0), (-7, -18) \mapsto (0, 1, 0, 0)$
L11n228	$(1, 2) \mapsto (0, 0, 1, 0), (-2, -4) \mapsto (0, 0, 1, 0), (0, 0) \mapsto (0, 1, 0, 0), (-3, -6) \mapsto (0, 1, 0, 0)$
L11n232	$(0, 2) \mapsto (0, 0, 1, 0)$
L11n233	$(-4, -14) \mapsto (0, 0, 1, 0), (-7, -20) \mapsto (0, 1, 0, 0)$
L11n235	$(-4, -10) \mapsto (0, 1, 0, 0), (-3, -8) \mapsto (0, 0, 1, 0), (-6, -14) \mapsto (0, 0, 0, 1)$
L11n236	$(-4, -12) \mapsto (0, 0, 1, 0)$
L11n239	$(-3, -4) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 1, 0, 0), (1, 4) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 1, 0, 0), (-4, -6) \mapsto$ $(0, 1, 0, 0), (0, 2) \mapsto (0, 1, 1, 0)$
L11n240	$(2, 8) \mapsto (0, 0, 1, 0), (-1, 2) \mapsto (0, 0, 1, 0), (1, 6) \mapsto (0, 1, 0, 0), (-2, 0) \mapsto (0, 1, 0, 0)$
L11n243	$(-2, -2) \mapsto (0, 1, 0, 0)$
L11n244	$(0, 0) \mapsto (0, 0, 1, 0)$
L11n252	$(-8, -18) \mapsto (0, 1, 0, 0), (-5, -12) \mapsto (0, 0, 1, 0)$
L11n253	$(-6, -16) \mapsto (0, 1, 0, 0), (-5, -14) \mapsto (0, 0, 1, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-8, -20) \mapsto (0, 0, 0, 1)$
L11n254	$(-4, -14) \mapsto (0, 0, 1, 0), (-7, -20) \mapsto (0, 1, 0, 0)$
L11n256	$(2, 7) \mapsto (0, 1, 0, 0), (-1, 1) \mapsto (0, 1, 0, 0), (1, 5) \mapsto (0, 0, 1, 0), (3, 9) \mapsto (0, 0, 1, 0), (-2, -1) \mapsto (0, 1, 0, 0),$ $(0, 3) \mapsto (0, 1, 1, 0)$
L11n257	$(-2, -5) \mapsto (0, 1, 0, 0), (-4, -9) \mapsto (0, 0, 1, 0), (-5, -11) \mapsto (0, 1, 0, 0)$
L11n261	$(4, 13) \mapsto (0, 1, 0, 0), (5, 15) \mapsto (0, 0, 1, 0), (2, 9) \mapsto (0, 1, 1, 0), (1, 7) \mapsto (0, 1, 0, 0)$
L11n262	$(-4, -11) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 0, 1, 0), (-7, -17) \mapsto (0, 1, 0, 0)$
L11n264	$(-4, -9) \mapsto (0, 0, 1, 0)$
L11n266	$(-3, -7) \mapsto (0, 1, 0, 0), (0, -1) \mapsto (0, 0, 1, 0)$
L11n267	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 3, 0, 0)$
L11n269	$(-2, -5) \mapsto (0, 0, 1, 0), (-3, -7) \mapsto (0, 1, 0, 0), (-4, -9) \mapsto (0, 0, 1, 0), (0, -1) \mapsto (0, 0, 1, 0), (-5, -11) \mapsto$ $(0, 1, 0, 0)$
L11n270	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L11n272	$(0, 1) \mapsto (0, 0, 1, 0), (-5, -9) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 1, 1, 0), (-1, -1) \mapsto (0, 0, 1, 0), (-4, -7) \mapsto$ $(0, 0, 1, 0), (-3, -5) \mapsto (0, 1, 0, 0)$
L11n273	$(-2, -3) \mapsto (0, 1, 0, 0)$
L11n274	$(-2, -5) \mapsto (0, 1, 0, 0), (-3, -7) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 1, 0), (-6, -13) \mapsto (0, 0, 1, 0), (-1, -3) \mapsto$ $(0, 0, 1, 0)$
L11n276	$(-7, -19) \mapsto (0, 1, 0, 0), (-4, -13) \mapsto (0, 0, 1, 0)$
L11n278	$(0, 1) \mapsto (0, 1, 0, 0), (1, 3) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 1, 1, 0)$
L11n280	$(-2, -3) \mapsto (0, 1, 0, 0)$
L11n284	$(0, 5) \mapsto (0, 1, 0, 0), (2, 9) \mapsto (0, 1, 0, 0)$
L11n285	$(-2, -5) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 0, 1, 0)$
L11n287	$(0, 1) \mapsto (0, 0, 1, 0), (-5, -9) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 2, 0, 0), (-1, -1) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto$ $(0, 1, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0)$
L11n288	$(-4, -11) \mapsto (0, 0, 2, 0), (-2, -7) \mapsto (0, 0, 1, 0), (-5, -13) \mapsto (0, 1, 0, 0), (-6, -15) \mapsto (0, 0, 1, 0), (-7, -17) \mapsto$ $(0, 1, 0, 0)$

L11n292	$(0, 3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L11n293	$(2, 7) \mapsto (0, 0, 1, 0), (-1, 1) \mapsto (0, 0, 1, 0), (-3, -3) \mapsto (0, 1, 0, 0), (1, 5) \mapsto (0, 1, 1, 0), (-2, -1) \mapsto (0, 1, 1, 0), (0, 3) \mapsto (0, 1, 1, 0)$
L11n294	$(2, 11) \mapsto (0, 2, 0, 0), (0, 7) \mapsto (0, 1, 0, 0)$
L11n299	$(0, 1) \mapsto (0, 0, 1, 0), (-5, -9) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 2, 0, 0), (-1, -1) \mapsto (0, 1, 1, 0), (-4, -7) \mapsto (0, 2, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0)$
L11n302	$(2, 7) \mapsto (0, 0, 1, 0), (1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 1, 0), (-3, -3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 0, 1, 0)$
L11n303	$(-2, 1) \mapsto (0, 1, 0, 0), (1, 7) \mapsto (0, 0, 1, 0)$
L11n306	$(-4, -11) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 0, 1, 0)$
L11n308	$(2, 7) \mapsto (0, 1, 0, 0), (-1, 1) \mapsto (0, 1, 0, 0), (3, 9) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 1, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L11n311	$(-1, 1) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 0, 1, 0), (1, 5) \mapsto (0, 0, 1, 0), (-4, -5) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0), (0, 3) \mapsto (0, 0, 1, 0)$
L11n313	$(-9, -21) \mapsto (0, 1, 0, 0), (-5, -13) \mapsto (0, 0, 1, 0), (-4, -11) \mapsto (0, 0, 1, 0), (-8, -19) \mapsto (0, 1, 1, 0), (-6, -15) \mapsto (0, 1, 1, 0), (-7, -17) \mapsto (0, 1, 0, 0)$
L11n315	$(0, 1) \mapsto (0, 0, 1, 0), (-5, -9) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 1, 1, 0), (-1, -1) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto (0, 1, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0)$
L11n320	$(-2, -5) \mapsto (0, 1, 1, 0), (-5, -11) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 0, 2, 0), (-1, -3) \mapsto (0, 0, 1, 0), (-3, -7) \mapsto (0, 1, 0, 0), (-6, -13) \mapsto (0, 1, 0, 0)$
L11n323	$(-5, -13) \mapsto (0, 0, 1, 0), (-4, -11) \mapsto (0, 0, 2, 0), (-3, -9) \mapsto (0, 1, 0, 0), (-6, -15) \mapsto (0, 1, 1, 0), (-7, -17) \mapsto (0, 1, 0, 0), (-2, -7) \mapsto (0, 0, 1, 0)$
L11n326	$(0, 1) \mapsto (0, 0, 1, 0), (-4, -7) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 0, 1, 0)$
L11n328	$(0, 3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L11n329	$(-2, -5) \mapsto (0, 0, 1, 0), (-3, -7) \mapsto (0, 1, 0, 0), (-4, -9) \mapsto (0, 0, 1, 0), (0, -1) \mapsto (0, 0, 1, 0)$
L11n332	$(-4, -9) \mapsto (0, 0, 1, 0)$
L11n333	$(-2, -3) \mapsto (0, 1, 0, 0)$
L11n334	$(0, 1) \mapsto (0, 0, 1, 0), (-5, -9) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 1, 1, 0), (-4, -7) \mapsto (0, 1, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0)$
L11n336	$(0, 3) \mapsto (0, 0, 1, 0)$
L11n337	$(-4, -11) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 0, 1, 0), (-7, -17) \mapsto (0, 1, 0, 0)$
L11n338	$(-6, -11) \mapsto (0, 1, 0, 0), (-3, -5) \mapsto (0, 0, 1, 0)$
L11n339	$(-3, -7) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 1, 0), (-6, -13) \mapsto (0, 0, 1, 0), (-7, -15) \mapsto (0, 1, 0, 0)$
L11n342	$(1, 5) \mapsto (0, 0, 1, 0), (-2, -1) \mapsto (0, 3, 0, 0)$
L11n345	$(4, 15) \mapsto (0, 0, 0, 1), (2, 11) \mapsto (0, 2, 0, 0)$
L11n347	$(-3, -7) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 0, 0), (-6, -13) \mapsto (0, 1, 0, 0)$
L11n348	$(-2, -5) \mapsto (0, 0, 1, 0), (-7, -15) \mapsto (0, 1, 0, 0), (-5, -11) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 1, 0), (-3, -7) \mapsto (0, 1, 1, 0), (-6, -13) \mapsto (0, 1, 1, 0)$
L11n350	$(-1, 1) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 0, 1, 0), (1, 5) \mapsto (0, 0, 1, 0), (-4, -5) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0), (0, 3) \mapsto (0, 1, 1, 0)$
L11n352	$(-5, -9) \mapsto (0, 0, 1, 0), (-6, -11) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 1, 0, 0)$
L11n354	$(-2, -5) \mapsto (0, 0, 1, 0), (-7, -15) \mapsto (0, 1, 0, 0), (-5, -11) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 1, 0), (-3, -7) \mapsto (0, 1, 0, 0), (-6, -13) \mapsto (0, 1, 1, 0)$
L11n357	$(0, 3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L11n359	$(-1, 1) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 0, 1, 0), (1, 5) \mapsto (0, 0, 1, 0), (-4, -5) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0), (0, 3) \mapsto (0, 0, 1, 0)$
L11n360	$(0, 1) \mapsto (0, 0, 1, 0), (-5, -9) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 1, 1, 0), (-1, -1) \mapsto (0, 0, 1, 0)$
L11n363	$(-3, -7) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 0, 0), (-1, -3) \mapsto (0, 1, 0, 0), (0, -1) \mapsto (0, 0, 1, 0)$
L11n366	$(2, 11) \mapsto (0, 1, 0, 0)$
L11n367	$(0, 1) \mapsto (0, 0, 1, 0)$
L11n368	$(-2, -5) \mapsto (0, 0, 1, 0), (-7, -15) \mapsto (0, 1, 0, 0), (-5, -11) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 0, 1, 0), (-3, -7) \mapsto (0, 1, 0, 0), (-6, -13) \mapsto (0, 1, 1, 0)$
L11n374	$(-4, -9) \mapsto (0, 0, 1, 0)$
L11n375	$(-4, -11) \mapsto (0, 0, 1, 0), (-8, -19) \mapsto (0, 0, 1, 0), (-5, -13) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 1, 0, 0)$
L11n376	$(-4, -9) \mapsto (0, 0, 1, 0)$
L11n379	$(-2, -5) \mapsto (0, 0, 1, 0), (0, -1) \mapsto (0, 0, 1, 0), (-3, -7) \mapsto (0, 1, 0, 0)$

L11n380	$(-2, -5) \mapsto (0, 1, 0, 0), (-4, -9) \mapsto (0, 0, 1, 0), (-6, -13) \mapsto (0, 1, 0, 0), (-5, -11) \mapsto (0, 1, 0, 0)$
L11n381	$(-5, -9) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 1, 0, 0)$
L11n383	$(0, 1) \mapsto (0, 0, 1, 0), (-5, -9) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 2, 0, 0), (-1, -1) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto (0, 1, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0)$
L11n385	$(-5, -13) \mapsto (0, 0, 1, 0), (-4, -11) \mapsto (0, 0, 2, 0), (-3, -9) \mapsto (0, 1, 0, 0), (-6, -15) \mapsto (0, 1, 1, 0), (-7, -17) \mapsto (0, 1, 0, 0), (-2, -7) \mapsto (0, 0, 1, 0)$
L11n387	$(-2, -3) \mapsto (0, 1, 0, 0)$
L11n390	$(0, -1) \mapsto (0, 0, 1, 0), (-3, -7) \mapsto (0, 1, 0, 0)$
L11n391	$(-3, -7) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 1, 0), (-6, -13) \mapsto (0, 0, 1, 0), (-7, -15) \mapsto (0, 1, 0, 0)$
L11n393	$(0, 1) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0), (-4, -7) \mapsto (0, 1, 0, 0), (-2, -3) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 1, 0, 0)$
L11n394	$(-4, -7) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 0, 1, 0)$
L11n395	$(1, 5) \mapsto (0, 0, 1, 0), (0, 3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L11n396	$(-2, -1) \mapsto (0, 1, 0, 0)$
L11n397	$(-1, 1) \mapsto (0, 1, 0, 0), (-3, -3) \mapsto (0, 0, 1, 0), (1, 5) \mapsto (0, 0, 1, 0), (-4, -5) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 2, 0, 0), (0, 3) \mapsto (0, 0, 1, 0)$
L11n398	$(-4, -11) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 0, 1, 0), (-7, -17) \mapsto (0, 1, 0, 0)$
L11n399	$(-3, -7) \mapsto (0, 0, 1, 0), (-4, -9) \mapsto (0, 1, 1, 0), (-6, -13) \mapsto (0, 0, 1, 0), (-7, -15) \mapsto (0, 1, 0, 0)$
L11n400	$(-6, -11) \mapsto (0, 1, 0, 0), (-3, -5) \mapsto (0, 0, 1, 0)$
L11n403	$(1, 5) \mapsto (0, 0, 1, 0), (-2, -1) \mapsto (0, 3, 0, 0)$
L11n404	$(-2, -5) \mapsto (0, 0, 1, 0), (-3, -7) \mapsto (0, 1, 0, 0), (0, -1) \mapsto (0, 0, 1, 0)$
L11n406	$(0, 1) \mapsto (0, 0, 1, 0)$
L11n408	$(0, 1) \mapsto (0, 0, 1, 0), (-3, -5) \mapsto (0, 0, 1, 0), (-4, -7) \mapsto (0, 1, 0, 0), (-1, -1) \mapsto (0, 1, 0, 0)$
L11n411	$(-9, -21) \mapsto (0, 1, 0, 0), (-8, -19) \mapsto (0, 0, 1, 0), (-6, -15) \mapsto (0, 1, 0, 0)$
L11n413	$(-1, 1) \mapsto (0, 0, 1, 0), (-4, -5) \mapsto (0, 1, 0, 0)$
L11n414	$(0, 1) \mapsto (0, 0, 1, 0), (-5, -9) \mapsto (0, 1, 0, 0), (-4, -7) \mapsto (0, 0, 1, 0), (-2, -3) \mapsto (0, 1, 1, 0), (-1, -1) \mapsto (0, 0, 1, 0)$
L11n417	$(0, 5) \mapsto (0, 1, 0, 0), (2, 9) \mapsto (0, 1, 0, 0)$
L11n420	$(-4, -11) \mapsto (0, 0, 1, 0), (-2, -7) \mapsto (0, 0, 1, 0)$
L11n423	$(0, 3) \mapsto (0, 1, 0, 0), (-2, -1) \mapsto (0, 1, 0, 0)$
L11n426	$(-4, -5) \mapsto (0, 1, 0, 0)$
L11n433	$(2, 11) \mapsto (0, 1, 0, 0), (0, 7) \mapsto (0, 1, 0, 0)$
L11n436	$(0, 1) \mapsto (0, 0, 1, 0)$
L11n439	$(-3, -4) \mapsto (0, 1, 0, 0), (0, 2) \mapsto (0, 0, 4, 0), (-2, -2) \mapsto (0, 0, 1, 0)$
L11n440	$(1, 6) \mapsto (0, 0, 1, 0), (-2, 0) \mapsto (0, 1, 0, 0), (0, 4) \mapsto (0, 1, 0, 0)$
L11n441	$(-2, -4) \mapsto (0, 1, 0, 0)$
L11n443	$(-3, -4) \mapsto (0, 1, 0, 0), (2, 6) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 0, 1, 0), (1, 4) \mapsto (0, 1, 0, 0), (-2, -2) \mapsto (0, 1, 1, 0), (0, 2) \mapsto (0, 1, 1, 0)$
L11n444	$(-2, -4) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 0, 0)$
L11n445	$(0, 4) \mapsto (0, 0, 1, 0)$
L11n446	$(-3, -4) \mapsto (0, 1, 0, 0), (2, 6) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 0, 1, 0), (1, 4) \mapsto (0, 1, 0, 0), (-2, -2) \mapsto (0, 1, 1, 0), (0, 2) \mapsto (0, 0, 2, 0)$
L11n447	$(-2, -4) \mapsto (0, 1, 0, 0)$
L11n448	$(-4, -6) \mapsto (0, 0, 1, 0), (-1, 0) \mapsto (0, 0, 1, 0), (0, 2) \mapsto (0, 0, 2, 0), (-2, -2) \mapsto (0, 1, 0, 0)$
L11n449	$(-2, -4) \mapsto (0, 1, 0, 0)$
L11n451	$(-2, -4) \mapsto (0, 1, 0, 0), (0, 0) \mapsto (0, 0, 1, 0), (-3, -6) \mapsto (0, 0, 1, 0), (-1, -2) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 0, 0)$
L11n452	$(0, 2) \mapsto (0, 1, 0, 0), (1, 4) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 4, 0, 0)$
L11n453	$(-6, -16) \mapsto (0, 0, 2, 0), (-4, -12) \mapsto (0, 0, 1, 0), (-8, -20) \mapsto (0, 0, 1, 0)$
L11n455	$(0, 2) \mapsto (0, 0, 1, 0), (-2, -2) \mapsto (0, 0, 1, 0)$
L11n456	$(-2, -4) \mapsto (0, 0, 1, 0), (0, 0) \mapsto (0, 0, 1, 0), (-3, -6) \mapsto (0, 1, 0, 0), (-4, -8) \mapsto (0, 1, 0, 0)$
L11n459	$(-2, 0) \mapsto (0, 1, 0, 0), (0, 4) \mapsto (0, 1, 0, 0)$

APPENDIX A. AN EXECUTIVE SUMMARY AND AN EXAMPLE

For the reader's convenience, we present a brief description of how to compute Sq^1 and Sq^2 .

- (1) $KG^{i,j}$ is the set of all Khovanov generators in bigrading (i, j) . The Khovanov chain group $KC^{i,j}$ in bigrading (i, j) is the \mathbb{F}_2 vector space with basis $KG^{i,j}$; for $\mathbf{x} \in KG^{i,j}$, and $\mathbf{c} \in KC^{i,j}$, we say $\mathbf{x} \in \mathbf{c}$ if the coefficient of \mathbf{x} in \mathbf{c} is 1, and $\mathbf{x} \notin \mathbf{c}$ otherwise. The Khovanov differential δ maps $KC^{i,j} \rightarrow KC^{i+1,j}$.
- (2) The Khovanov homology in bigrading (i, j) is denoted $Kh^{i,j}$. For a cycle $\mathbf{c} \in KC^{i,j}$, $[\mathbf{c}] \in Kh^{i,j}$ denotes the corresponding homology element.
- (3) Let n be the number of crossings in the oriented link diagram. Then there is a natural map $\mathcal{F}: KG = \coprod_{i,j} KG^{i,j} \rightarrow \{0, 1\}^n$.
- (4) For $u, v \in \{0, 1\}^n$ with $u = (\epsilon_1, \dots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \dots, \epsilon_n)$ and $v = (\epsilon_1, \dots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \dots, \epsilon_n)$, define

$$s(\mathcal{C}_{u,v}) = (\epsilon_1 + \dots + \epsilon_{i-1}) \pmod{2} \in \mathbb{F}_2.$$

For $u, v \in \{0, 1\}^n$ with $u = (\epsilon_1, \dots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \dots, \epsilon_{j-1}, 1, \epsilon_{j+1}, \dots, \epsilon_n)$ and $v = (\epsilon_1, \dots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \dots, \epsilon_{j-1}, 0, \epsilon_{j+1}, \dots, \epsilon_n)$, define

$$f(\mathcal{C}_{u,v}) = (\epsilon_1 + \dots + \epsilon_{i-1})(\epsilon_{i+1} + \dots + \epsilon_{j-1}) \pmod{2} \in \mathbb{F}_2.$$

- (5) For $\mathbf{x} \in KG^{i+2,j}$ and $\mathbf{y} \in KG^{i,j}$, let

$$\mathcal{G}_{\mathbf{x},\mathbf{y}} = \{\mathbf{z} \in KG^{i+1,j} \mid \mathbf{x} \in \delta\mathbf{z}, \mathbf{z} \in \delta\mathbf{y}\}.$$

The number of elements in $\mathcal{G}_{\mathbf{x},\mathbf{y}}$ is 0, 2 or 4. The ladybug matching \mathfrak{l} is a well-defined collection $\{\mathfrak{l}_{\mathbf{x},\mathbf{y}}\}$ of fixed point free involutions of $\mathcal{G}_{\mathbf{x},\mathbf{y}}$. The only case of interest is when $\mathcal{G}_{\mathbf{x},\mathbf{y}}$ has 4 elements; the ladybug matching in that case is shown in [Figure 2.1b](#).

- (6) Let $\mathbf{c} \in KC^{i,j}$ be a cycle. For $\mathbf{x} \in KG^{i+1,j}$, define

$$\mathcal{G}_{\mathbf{c}}(\mathbf{x}) = \{\mathbf{y} \in KG^{i,j} \mid \mathbf{x} \in \delta\mathbf{y}, \mathbf{y} \in \mathbf{c}\}.$$

For $\mathbf{x} \in KG^{i+2,j}$, define

$$\mathcal{G}_{\mathbf{c}}(\mathbf{x}) = \{(\mathbf{z}, \mathbf{y}) \in KG^{i+1,j} \times KG^{i,j} \mid \mathbf{x} \in \delta\mathbf{z}, \mathbf{z} \in \delta\mathbf{y}, \mathbf{y} \in \mathbf{c}\}.$$

- (7) For $\mathbf{c} \in KC^{i,j}$ a cycle, a boundary matching \mathbf{m} for \mathbf{c} is a collection of pairs $(\mathbf{b}_{\mathbf{x}}, \mathbf{s}_{\mathbf{x}})$, one for each $\mathbf{x} \in KG^{i+1,j}$, where:
 - $\mathbf{b}_{\mathbf{x}}$ is a fixed point free involution of $\mathcal{G}_{\mathbf{c}}(\mathbf{x})$, and
 - $\mathbf{s}_{\mathbf{x}}$ is a map $\mathcal{G}_{\mathbf{c}}(\mathbf{x}) \rightarrow \mathbb{F}_2$, such that for all $\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{x})$,

$$\{\mathbf{s}_{\mathbf{x}}(\mathbf{y}), \mathbf{s}_{\mathbf{x}}(\mathbf{b}_{\mathbf{x}}(\mathbf{y}))\} = \begin{cases} \{0, 1\} & \text{if } s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{y})}) = s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{b}_{\mathbf{x}}(\mathbf{y}))}) \\ \{0\} & \text{otherwise.} \end{cases}$$

- (8) Given a cycle $\mathbf{c} \in KC^{i,j}$, any boundary matching $\mathbf{m} = \{(\mathbf{b}_x, \mathbf{s}_x)\}$ for \mathbf{c} , and $\mathbf{x} \in KG^{i+2,j}$, consider the edge-labeled graph $\mathfrak{G}_c(\mathbf{x})$, whose vertices are the elements of $\mathcal{G}_c(\mathbf{x})$ and whose edges are the following.

- There is an unoriented edge between (\mathbf{z}, \mathbf{y}) and $(\mathbf{z}', \mathbf{y})$, if the ladybug matching $\mathbf{l}_{\mathbf{x}, \mathbf{y}}$ matches \mathbf{z} and \mathbf{z}' . This edge is labeled by $f(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{y})})$.
- There is an edge between (\mathbf{z}, \mathbf{y}) and $(\mathbf{z}, \mathbf{y}')$ if the matching \mathbf{b}_z matches \mathbf{y} with \mathbf{y}' . This edge is labeled by 0. Furthermore, if $\mathbf{s}_z(\mathbf{y}) = 0$ and $\mathbf{s}_z(\mathbf{y}') = 1$, then this edge is oriented from (\mathbf{z}, \mathbf{y}) to $(\mathbf{z}, \mathbf{y}')$; if $\mathbf{s}_z(\mathbf{y}) = 1$ and $\mathbf{s}_z(\mathbf{y}') = 0$, then this edge is oriented from $(\mathbf{z}, \mathbf{y}')$ to (\mathbf{z}, \mathbf{y}) ; and if $\mathbf{s}_z(\mathbf{y}) = \mathbf{s}_z(\mathbf{y}')$, then the edge is unoriented.

Each component of $\mathfrak{G}_c(\mathbf{x})$ is a cycle. Let $\#|\mathfrak{G}_c(\mathbf{x})|$ denote the number of components of the graph modulo 2. Let $f(\mathfrak{G}_c(\mathbf{x})) \in \mathbb{F}_2$ be the sum of all the edge-labels in the graph. Partition the oriented edges of $\mathfrak{G}_c(\mathbf{x})$ into two sets, such that if two edges from the same cycle are in the same set, they are oriented in the same direction; then $g(\mathfrak{G}_c(\mathbf{x}))$ is the number modulo 2 of the elements in either set.

- (9) Let $\mathbf{c} \in KC^{i,j}$ be a cycle. Choose any boundary matching $\mathbf{m} = \{(\mathbf{b}_x, \mathbf{s}_x)\}$ for \mathbf{c} . Define the chains $\text{sq}_{\mathbf{m}}^1(\mathbf{c}) \in KC^{i+1,j}$ and $\text{sq}_{\mathbf{m}}^2(\mathbf{c}) \in KC^{i+2,j}$ as

$$\begin{aligned} \text{sq}_{\mathbf{m}}^1(\mathbf{c}) &= \sum_{\mathbf{x} \in KG^{i+1,j}} \left(\sum_{\mathbf{y} \in \mathcal{G}_c(\mathbf{x})} \mathbf{s}_x(\mathbf{y}) \right) \mathbf{x}, \\ \text{sq}_{\mathbf{m}}^2(\mathbf{c}) &= \sum_{\mathbf{x} \in KG^{i+2,j}} \left(\#|\mathfrak{G}_c(\mathbf{x})| + f(\mathfrak{G}_c(\mathbf{x})) + g(\mathfrak{G}_c(\mathbf{x})) \right) \mathbf{x}. \end{aligned}$$

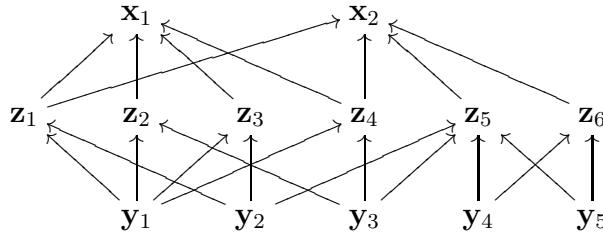
Then $\text{sq}_{\mathbf{m}}^1(\mathbf{c})$ and $\text{sq}_{\mathbf{m}}^2(\mathbf{c})$ are cycles; and

$$[\text{sq}_{\mathbf{m}}^1(\mathbf{c})] = \text{Sq}^1([\mathbf{c}]),$$

$$[\text{sq}_{\mathbf{m}}^2(\mathbf{c})] = \text{Sq}^2([\mathbf{c}]).$$

The following is an artificial example to illustrate [Definition 2.5](#) and [Definition 2.8](#).

Example A.1. Assume $KG^{i,j} = \{\mathbf{y}_1, \dots, \mathbf{y}_5\}$, $KG^{i+1,j} = \{\mathbf{z}_1, \dots, \mathbf{z}_6\}$ and $KG^{i+2,j} = \{\mathbf{x}_1, \mathbf{x}_2\}$, and the Khovanov differential δ has the following form.



Assume that the sign assignment and the frame assignment are as follows.

(\mathbf{x}, \mathbf{y})	$s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{y})})$	(\mathbf{x}, \mathbf{y})	$s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{y})})$	(\mathbf{x}, \mathbf{y})	$s(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{y})})$	(\mathbf{x}, \mathbf{y})	$f(\mathcal{C}_{\mathcal{F}(\mathbf{x}), \mathcal{F}(\mathbf{y})})$
$(\mathbf{z}_1, \mathbf{y}_1)$	1	$(\mathbf{z}_4, \mathbf{y}_3)$	1	$(\mathbf{x}_1, \mathbf{z}_3)$	0	$(\mathbf{x}_1, \mathbf{y}_1)$	1
$(\mathbf{z}_2, \mathbf{y}_1)$	1	$(\mathbf{z}_5, \mathbf{y}_3)$	0	$(\mathbf{x}_2, \mathbf{z}_3)$	0	$(\mathbf{x}_2, \mathbf{y}_1)$	0
$(\mathbf{z}_3, \mathbf{y}_1)$	0	$(\mathbf{z}_5, \mathbf{y}_4)$	0	$(\mathbf{x}_1, \mathbf{z}_4)$	0	$(\mathbf{x}_1, \mathbf{y}_2)$	0
$(\mathbf{z}_4, \mathbf{y}_1)$	0	$(\mathbf{z}_6, \mathbf{y}_4)$	0	$(\mathbf{x}_2, \mathbf{z}_4)$	0	$(\mathbf{x}_2, \mathbf{y}_2)$	1
$(\mathbf{z}_1, \mathbf{y}_2)$	1	$(\mathbf{z}_5, \mathbf{y}_5)$	0	$(\mathbf{x}_2, \mathbf{z}_5)$	0	$(\mathbf{x}_1, \mathbf{y}_3)$	1
$(\mathbf{z}_3, \mathbf{y}_2)$	0	$(\mathbf{z}_6, \mathbf{y}_5)$	0	$(\mathbf{x}_2, \mathbf{z}_6)$	1	$(\mathbf{x}_2, \mathbf{y}_3)$	0
$(\mathbf{z}_5, \mathbf{y}_2)$	0	$(\mathbf{x}_1, \mathbf{z}_1)$	0			$(\mathbf{x}_2, \mathbf{y}_4)$	1
$(\mathbf{z}_2, \mathbf{y}_3)$	0	$(\mathbf{x}_1, \mathbf{z}_2)$	0			$(\mathbf{x}_2, \mathbf{y}_5)$	1

Finally, assume that the ladybug matching $\mathbf{l}_{\mathbf{x}_1, \mathbf{y}_1}$ matches \mathbf{z}_1 with \mathbf{z}_4 and \mathbf{z}_2 with \mathbf{z}_3 .

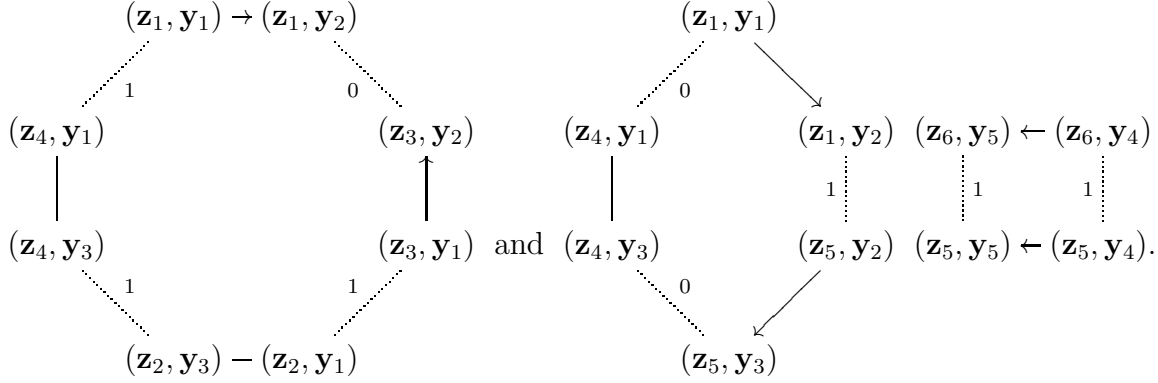
Let us start with the cycle $\mathbf{c} \in KC^{i,j}$ given by $\mathbf{c} = \sum_{i=1}^5 \mathbf{y}_i$. In order to compute $\text{Sq}^1(\mathbf{c})$ and $\text{Sq}^2(\mathbf{c})$, we need to choose a boundary matching $\mathbf{m} = \{(\mathbf{b}_{\mathbf{z}_j}, \mathbf{s}_{\mathbf{z}_j})\}$ for \mathbf{c} . Let us choose the following boundary matching.

j	$\mathbf{b}_{\mathbf{z}_j}$	$\mathbf{s}_{\mathbf{z}_j}$	j	$\mathbf{b}_{\mathbf{z}_j}$	$\mathbf{s}_{\mathbf{z}_j}$
1	$\mathbf{y}_1 \leftrightarrow \mathbf{y}_2$	$\mathbf{y}_1 \rightarrow 0, \mathbf{y}_2 \rightarrow 1$	4	$\mathbf{y}_1 \leftrightarrow \mathbf{y}_3$	$\mathbf{y}_1, \mathbf{y}_3 \rightarrow 0$
2	$\mathbf{y}_1 \leftrightarrow \mathbf{y}_3$	$\mathbf{y}_1, \mathbf{y}_3 \rightarrow 0$	5	$\mathbf{y}_2 \leftrightarrow \mathbf{y}_3, \mathbf{y}_4 \leftrightarrow \mathbf{y}_5$	$\mathbf{y}_2, \mathbf{y}_4 \rightarrow 0, \mathbf{y}_3, \mathbf{y}_5 \rightarrow 1$
3	$\mathbf{y}_1 \leftrightarrow \mathbf{y}_2$	$\mathbf{y}_1 \rightarrow 0, \mathbf{y}_2 \rightarrow 1$	6	$\mathbf{y}_4 \leftrightarrow \mathbf{y}_5$	$\mathbf{y}_4 \rightarrow 0, \mathbf{y}_5 \rightarrow 1$

Then, the cycle $\text{sq}_{\mathbf{m}}^1(\mathbf{c})$ is given by

$$\begin{aligned}
\text{sq}_{\mathbf{m}}^1(\mathbf{c}) &= \sum_{j=1}^6 \left(\sum_{\mathbf{y} \in \mathcal{G}_{\mathbf{c}}(\mathbf{z}_j)} \mathbf{s}_{\mathbf{z}_j}(\mathbf{y}) \right) \mathbf{z}_j \\
&= (\mathbf{s}_{\mathbf{z}_1}(\mathbf{y}_1) + \mathbf{s}_{\mathbf{z}_1}(\mathbf{y}_2)) \mathbf{z}_1 + (\mathbf{s}_{\mathbf{z}_2}(\mathbf{y}_1) + \mathbf{s}_{\mathbf{z}_2}(\mathbf{y}_3)) \mathbf{z}_2 + (\mathbf{s}_{\mathbf{z}_3}(\mathbf{y}_1) + \mathbf{s}_{\mathbf{z}_3}(\mathbf{y}_2)) \mathbf{z}_3 \\
&\quad + (\mathbf{s}_{\mathbf{z}_4}(\mathbf{y}_1) + \mathbf{s}_{\mathbf{z}_4}(\mathbf{y}_3)) \mathbf{z}_4 + (\mathbf{s}_{\mathbf{z}_5}(\mathbf{y}_2) + \mathbf{s}_{\mathbf{z}_5}(\mathbf{y}_3) + \mathbf{s}_{\mathbf{z}_5}(\mathbf{y}_4) + \mathbf{s}_{\mathbf{z}_5}(\mathbf{y}_5)) \mathbf{z}_5 \\
&\quad + (\mathbf{s}_{\mathbf{z}_6}(\mathbf{y}_4) + \mathbf{s}_{\mathbf{z}_6}(\mathbf{y}_5)) \mathbf{z}_6 \\
&= \mathbf{z}_1 + \mathbf{z}_3 + \mathbf{z}_6.
\end{aligned}$$

In order to compute $\text{sq}_{\text{m}}^2(\mathbf{c})$, we need to study the graphs $\mathfrak{G}_{\mathbf{c}}(\mathbf{x}_1)$ and $\mathfrak{G}_{\mathbf{c}}(\mathbf{x}_2)$, which are the following:



The Type (e-1) edges are represented by the dotted lines; they are unoriented and are labeled by elements of \mathbb{F}_2 . The Type (e-2) edges are represented by the solid lines; they are labeled by 0 and are sometimes oriented. Therefore, the cycle $\text{sq}_{\text{m}}^2(\mathbf{c})$ is given by

$$\begin{aligned} \text{sq}_{\text{m}}^2(\mathbf{c}) &= \sum_{j=1}^2 \left(\#|\mathfrak{G}_{\mathbf{c}}(\mathbf{x}_j)| + f(\mathfrak{G}_{\mathbf{c}}(\mathbf{x}_j)) + g(\mathfrak{G}_{\mathbf{c}}(\mathbf{x}_j)) \right) \mathbf{x}_j \\ &= (1 + 1 + 1)\mathbf{x}_1 + (0 + 1 + 1)\mathbf{x}_2 \\ &= \mathbf{x}_1. \end{aligned}$$

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